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Dedicated to my parents, Perihan and Cafer Dur.

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Essays on Real Life Allocation Problems

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In the first chapter, we introduce a new matching model to mimic inter-college tuition exchange programs for dependents of faculty to attend other colleges tuition-free. Each participating college has to avoid being a net-exporter of students. Programs use decentralized markets making it difficult to achieve balance. We show that stable equilibria discourage net-exporting colleges from exchange. We introduce two-sided top-trading-cycles (2S-TTC) mechanism that is balanced-efficient, student-strategy-proof, and respecting priority bylaws regarding dependent eligibility. Moreover, it encourages exchange, since full participation is dominant strategy for colleges. We prove 2S-TTC is the unique mechanism fulfilling these objectives and introduce new student-strategy-proof mechanisms to achieve other objectives.

In the second chapter, we consider a house allocation with existing tenants model in which each transaction is costly for the central authority, a housing office. We compare two widely studied mechanisms, deferred acceptance (DA) and top trading cycles (TTC), based on their costs for the housing

offices. A mechanism in which more existing tenants are assigned to their current house is preferred for the housing offices due to the costs of moving. We show that although there is no dominance between the two mechanisms, DA has more desirable features in terms of the cost efficiency for the housing offices. Then we include the welfare of the housing office in the welfare analysis and redefine the Pareto efficiency notion. We show that every fair matching is Pareto efficient. Based on the extended Pareto efficiency definition, the DA mechanism is the unique Pareto efficient, fair, and strategy-proof mechanism.

Finally, the third chapter characterizes the top trading cycles mechanism for the school choice problem. Schools may have multiple available seats to be assigned to students. For each school a strict priority ordering of students is determined by the school district. Each student has strict preference over the schools. We first define weaker forms of fairness, consistency and resource monotonicity. We show that the top trading cycles mechanism is the unique Pareto efficient and strategy-proof mechanism that satisfies the weaker forms of fairness, consistency and resource monotonicity. To our knowledge this is the first axiomatic approach to the top trading cycles mechanism in the school choice problem where schools have a capacity greater than one.

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Chapter 1

Tuition Exchange

[1]

1.1 Introduction

It has always been difficult for small colleges and universities to compete with bigger schools in hiring the best and brightest faculty. Colleges located farther away from major metropolitan areas face a similar challenge. Tuition exchange programs play a prominent role for these colleges in attracting and retaining highly qualified faculty.[2]

In a tuition exchange program, qualified dependents of faculty and staff are given tuition–waivers at their home institutions, and can swap these waivers with dependents of faculty and staff at other colleges with the condition of mutual acceptance of dependents to these institutions. Tuition exchange has become a desirable benefit that adds value to an attractive employment

[1] This essay is drawn from the joint work with Utku Ünver.

[2] "Tuition Exchange enables us to compete with the many larger institutions in our area for talented faculty and staff. The generous awards help us attract and retain employees, especially in high-demand fields like nursing and IT." – Frank Greco, Director of Human Resources, Chatham University, from the home page of The Tuition Exchange, Inc., www.tuitionexchange.org, retrieved on 09/19/2012.

package without creating additional out–of–pocket expenses for colleges.

One of the prominent programs is "The Tuition Exchange, Inc." (TTEI),[3] which is also the oldest and largest of its kind in the US. TTEI is a reciprocal scholarship program for children (and other family members) of faculty and staff employed at more than 600 participating institutions. Member colleges are spread over 47 states and the District of Columbia. Both research universities and liberal arts colleges are members. *US News and World Report* lists 38 member colleges in the best 200 research universities and 46 member colleges in the best 100 liberal arts colleges. Every year an average of 20 new institutions join the program. Through TTEI, on average, 6,000 scholarships are awarded annually, with amounts averaging about $24,000. Despite TTEI's large volume, other tuition exchange programs clear more than 50% of all exchange transactions in the US.[4]

Each participating college establishes its own policies and procedures for determining the eligibility of employees' dependents for exchange and the number of scholarships it will grant each year. Each member college has agreed to maintain a balance between the number of students sponsored by that institution ("exports") and the number of scholarships awarded to students sponsored by other member colleges who will enroll at that institution ("imports"). Every year, colleges aim to maintain a one–to–one balance between the num-

[3] See http://www.tuitionexchange.org.

[4] An alternative to tuition exchange is monetary subsidization of faculty members. Most colleges do not prefer this, as any direct compensation is taxable income, whereas a tuition exchange scholarship is not.

ber of exports and imports. In particular, if the number of exports exceeds the number of imports, that college may be suspended from TTEI. Therefore, each college determines the (maximum) number of students it will sponsor according the expected number of students who will apply to that institution for a tuition scholarship. In order not to be suspended from TTEI, colleges often set the maximum number of sponsored students in a precautionary manner. Many colleges explicitly mention in their application documents that in order to guarantee their continuation in the program they need to limit the number of sponsored students.[5] As a result, in many cases not all qualified dependents are sponsored. Colleges often use the length of the related employee's tenure to prioritize the eligible students. Academic achievements are not necessarily relevant to internal priorities, so that academically less successful students may be prioritized over more successful ones.

From a market and efficiency perspective, tuition benefits at home institutions are distortionary as they cause the dependents to attend their home institutions. Tuition exchange programs help this distortion to be corrected. Therefore, making the clearinghouses employed by these programs as efficient as possible would minimize this distortion. Our goal in this paper is to show the problems with the current exchange systems and propose better mechanisms to improve efficiency.

First, we introduce a new class of two–sided matching problems to

[5] Lafayette College, Daemen College, DePaul University, and Lewis University are just a few examples.

model the matching process within a tuition exchange program. In a tuition exchange problem, eligible dependents of faculty members are prioritized according to the length of faculty tenure or some other external criterion. Each college then determines its quota, which is the maximum number of students it will sponsor (its "export eligibility quota") and the maximum number it will admit (its "import quota") through the program. Then, the sponsored students are awarded with scholarships or remain unmatched according to the preferences of colleges over sponsored students and the preferences of sponsored students over colleges subject to the quota constraints.[6] Students who are not sponsored by their home colleges do not receive scholarships, and hence cannot participate in the program.

We first show that there may not exist a balanced[7] matching that is also non-wasteful[8] and individually rational[9] (Proposition 4). This fact is

[6]This class of problems is closely related to the well–known college admissions problem of Gale and Shapley (1962). From a theoretical point of view, there are two important differences between the tuition exchange problem and the college–admissions problem. In the college admissions problem, all students are considered eligible, whereas in the tuition exchange problem the set of eligible students is determined according to the internal priority order of each college. Hence, in college admissions, colleges determine the quota of students they will accept, while in tuition exchange, they set the quota of internal students they will sponsor in addition to the import quota. Secondly, in tuition exchange, maintaining a one–to–one balance between the exports and imports is the central issue for colleges. However, in college admissions, the students are not sponsored by any college before the enrollment, and hence, *balancedness* is not a concern for colleges.

[7]A matching is *balanced* if each college maintains a balance between the number of students sponsored by that institution ("exports") and the number of scholarships awarded to students sponsored by other member colleges enrolling at that institution ("imports").

[8]A matching is *non-wasteful* if, whenever a college has an open slot, either each student prefers her assignment to that college or that college finds that student unacceptable.

[9]A matching is individually rational if each agent finds its/her all assigned partners acceptable.

caused by the non-wastefulness notion that respects exogenously set quotas. Because of the balancedness condition, this is not an appropriate property for our problem.

In the earlier two–sided matching literature, stability a la Gale and Shapley (1962) has been the central solution concept. Proposition 4 also implies that stability and balancedness are incompatible. We also show that there doesn't exist a weakly stable mechanism that always selects an undominated balanced (i.e., *balanced–efficient*) matching (Proposition 5). A matching is *weakly stable* if it is individually rational and there is no block of a college and a student such that the student prefers the college to her assignment and the college prefers the student to one of the students currently matched with it. Hence, this concept respects immunity against the mutually beneficial "replacement" of a student without changing the overall balance of the school.[10]

Moreover, in any stable mechanism outcome, we show that it is the best response for a college with a negative balance to decrease its export quota, and increasing export quota is never a best response (Theorem 1). Behaving with respect to such a best response may further cause another college to have a negative balance, as decrease in participation never improves the negative balances of other colleges (Theorem 2). Hence, if we take stability as a benchmark market equilibrium concept in a decentralized market, such an equilibrium, in general, discourages exchange and can prevent the market

[10]A weakly stable and balanced matching always exists; for example, the null matching is one.

from extracting the highest gains from trade.

On the other hand, in our proposed centralized balanced–efficient mechanism, colleges prefer to act non-strategically while determining their quotas (Theorem 5).

We then restrict our attention to the set of balanced–efficient mechanisms. Unfortunately, there exists no balanced–efficient, and individually rational mechanism that is incentive compatible for colleges (Proposition 8).

We propose a mechanism that is similar to Gale's top–trading–cycles (TTC) mechanism introduced by Shapley and Scarf (1974) for finding core and competitive allocations for a simple "house" exchange market without money. In the school choice problem (cf. Abdulkadiroğlu and Sönmez, 2003) and house allocation problem with existing agents (cf. Abdulkadiroğlu and Sönmez, 1999), variants of mechanisms related to Gale's TTC have been introduced and their properties extensively discussed. In all these problems, one side of the market is considered to be objects to be consumed and are not included in the welfare analysis. Schools and houses have no preferences, but "priorities." For instance, in the school choice problem, the priorities of schools over the students are determined according to test scores or proximity of their residential location to the school, whereas in the house allocation problem the priorities of houses over the students are determined by random draws or seniority. Hence, they are not strategic agents. In these markets, it is shown that TTC is individually rational, strategy-proof, and Pareto efficient. In tuition exchange, in contrast to school choice and house allocation, both sides of the market are strategic and should be included to the welfare analysis. To capture this difference, we modify the TTC mechanism and refer to it as the *two–sided top–trading–cycles* (2S-TTC) mechanism. To our knowledge, this is the first time that a variation of the TTC mechanism is used in a two–sided matching problem where both sides are strategic. We show that 2S-TTC has appealing properties. First of all, its outcome is balanced–efficient.[11] It is individually rational and cannot be manipulated by students; and it also respects the internal priorities used to determine the set of sponsored students by each college (Theorems 3 and 4). We also show that it is the unique mechanism satisfying these four properties (Theorem 6).[12] We also show the independence of the axioms, which are held only by 2S-TTC.

Although 2S-TTC is balanced–efficient, it may not match the maximum

[11]This result is not an extension of the classical Pareto efficiency result of TTC in a one–sided market. Here, colleges are players and they have multiple seats over which they have "responsive" preferences. Therefore, by assigning a college highly preferred students and also some unacceptable ones, an individually rational balanced matching can potentially be (weakly) improved for everyone, colleges and students alike, while violating individual rationality for colleges and yet still obeying balancedness. We show that it is not possible to improve upon 2S-TTC's outcome in such a fashion.

[12]Ma (1994) had previously characterized TTC when there is a single seat at each school through Pareto efficiency, individual rationality, and strategy-proofness for students. Our characterization uses a different proof technique not only from Ma (1994), but also subsequent simpler proofs of this prior result by Sönmez (1995) and Svensson (1999), which do not work in the current setup. There are a few other related characterization results in the literature: Abdulkadiroğlu and Che (2010) characterize school choice TTC a la Abdulkadiroğlu and Sönmez (2003); Pycia and Ünver (2011a) characterize general individually rational TTC rules a la Pápai (2000) when there are more objects than agents; and Sönmez and Ünver (2010) characterize TTC rules a la Abdulkadiroğlu and Sönmez (1999) for house allocation with existing tenants. Besides these characterizations, a related mechanism to our 2S-TTC was proposed by Ekici (2011) in a one-sided matching problem for temporary house exchanges with unit quotas.

possible number of students while maintaining balancedness. Such a solution would maximize gains from trade for the tuition exchange program. We show that if the maximal balanced solution is different from the 2S-TTC outcome even for one preference profile, it can be manipulated by students (Proposition 11). Hence, we do not recommend such a solution.

Sometimes exact balancedness is not needed, and average balancedness over time may be all that is required. We also adapt 2S-TTC to such dynamic environments. For predetermined minimum and maximum cut-off values, we introduce the *two–sided top–trading–cycles–and–chains* mechanism *with tolerance* (2S-TTCC) and show that it is strategy–proof for students, individually rational, respecting internal priorities, and Pareto undominated among all individually rational matchings in the same permissible imbalance interval (see Appendix A.4). The minimum and maximum imbalance values can be made state–dependent and optimized dynamically to obtain average balancedness over time for each college.

There also exist tuition exchange co-ops where balancedness is not the participants' first concern. For these programs, we also introduce a mechanism that takes stability as the primary constraint. This mechanism is based on the student–proposing deferred–acceptance (DA) algorithm of Gale and Shapley (1962). We repeatedly apply DA starting from a market in which import quotas of schools are equal to their export quotas. After the first execution, if an unbalanced matching is found, we increase the eligible export quota of a college that runs a positive balance (with more imports than exports) by one if possible. We execute DA again, re-adjust the eligible export quota of one college, and repeat the above procedure. We continue this procedure. We show that eventually the outcome of this mechanism converges to a stable matching of a particular market with the least possible aggregate imbalance for colleges (Proposition 14). Moreover, the outcome is independent of the order in which we include new students (Proposition 13). That is, each repetition improves the aggregate balance of the schools. We refer to this rule as the repeated deferred–acceptance (RDA) mechanism. We also prove that this mechanism is strategy-proof for students (Theorem 7). [13]

We also analyze tuition exchange problems while assuming that colleges have preferences over their outgoing students. In this case, under very mild assumptions, we show that the results we found earlier for colleges with preferences over incoming students hold for 2S-TTC regarding immunity against quota manipulation and characterization through the aforementioned four properties. Hence, 2S-TTC's nice properties are robust to the specification of college preferences.

[13] The strategy-proofness of RDA is somewhat surprising, because by manipulating her preferences under RDA, a student can potentially change the export quota of a school, and hence, the set of students participating in the exchange. Although the strategy-proofness result is in the flavor of the results of Dubins and Freedman (1981) and Roth (1982), which showed that DA mechanism is strategy-proof for students, our proof does not use these results.

1.2 Related Markets

Although tuition exchange is a two–sided matching market, students cannot participate in the market activity unless their home colleges sponsor them. Hence, an import/export balance emerges as an important feature of sustainable outcomes. This is the most important feature of tuition exchange that distinguishes it from previously studied matching markets. However, tuition exchange is not unique; there are numerous similar markets, which we briefly discuss here. What we learn in the current analysis can help us understand these other markets better as well.

Tuition exchange markets are closely related to favor exchange markets, also known as "time banks," where time spent doing a favor or the number of favors is used as the currency of exchange.[14] The currency of transactions responds to a positive imbalance in tuition exchange markets. Baby sitting co-ops are a leading example of such time banks. Negative–balance aversion as in tuition exchange markets is also known to affect such banks adversely and cause the markets to shut down. Parallels have been drawn between such markets and complex monetary systems, where liquidity shortage is known to cause recessions (cf. Sweeney and Sweeney, 1977; Krugman, 1998).

Balanced and efficient employee exchange within or across organizations is a direct application of our centralized market design using the 2S-TTC mechanism. Some examples are employee exchange among country headquar-

[14] See Mobius (2001) for a dynamic analysis of a favor exchange model.

ters of multinational firms; doctor and staff exchange among hospitals; faculty exchange among public health schools, and so on.

Secondary transfer window transactions in European club soccer have the same features as our tuition exchange markets: monetary negotiations no longer matter since a player is paid by his primary club regardless of which team he plays for; roster size restrictions dictate balancedness; club and player preferences dictate trades in this window. The window has a very limited duration, in which centralized mechanisms can be used for synchronized trades by UEFA.

Large appeals process transactions in centralized matching markets have tuition exchange features as well. For example, the high school appeals process of the New York City public school district has features similar to those of tuition exchange (cf. Abdulkadiroğlu, Pathak, and Roth, 2005).

The Erasmus student exchange program among universities in Europe is another example of a market to which we can apply our findings. In the past twenty years, over two million students have benefited from grant opportunities of the Erasmus exchange program. There are huge imbalances between the number of students exported and imported by each country. Moreover, countries with high positive balances are not willing to match the quota requests of the net exporter countries. This precautionary behavior may lead to inefficiencies as in tuition exchange markets (see Appendix A.2).

1.3 Tuition Exchange Programs

In this section, we provide a brief description of "The Tuition Exchange, Inc." (TTEI) program as a representative of similar programs.[15] In TTEI, every participating institution determines the number of outgoing students it can certify, as well as how many TTEI awards it will grant to incoming students each year. Each college determines its export and import quota. Then each faculty member submits the TTEI application to the registration office of their college. If the number of applicants is greater than the number of outgoing students that the college is willing to certify, then the college decides whom to certify based on years of service or some other criterion.

Each student certified eligible submits a list of colleges to the liaison office of her home institution. Each liaison office sends a copy of the TTEI "Certificate of Eligibility" to the TTEI liaison officer at the participating colleges and universities listed by the eligible dependents. Certification only means the student is eligible for a TTEI award; it is not a guarantee of an award. The eligible student must apply for admission to the college(s) in which she is interested, following each institution's application procedures and deadlines. After admission decisions have been made, the admissions offices or TTEI liaisons at her listed institutions inform her whether or not she will be offered a TTEI award. TTEI scholarships are competitive and some eligible applicants may not receive them. That is, the sponsoring institution cannot guarantee

[15] In Appendix A.1, we describe the features of the other tuition exchange programs.

that an "export" candidate, regardless of qualifications, will receive a TTEI scholarship. Institutions choose their scholarship recipients ("imports") based on applicants' academic profiles. Some export candidates may receive more than one scholarship offer. It is also possible that an export candidate may not be admitted to her listed colleges during the admission process, and as a result, she will not be considered for a scholarship. The procedures of TTEI applications and regular college admission in the U.S. have several common features, but they differ in that in tuition exchange, regular college admission is a prerequisite, and each participating student needs to be certified by her home college for eligibility. In the following examples, we illustrate the main distinctions between tuition exchange and regular college admission, and the main incentive problems faced by college tuition exchange liaison offices.

Example 1 Suppose there are three colleges in the Exchange: University of Southern California (USC), University of Pittsburgh (Pitt), and Boston University (BU). The parents of Selma and Clay work at USC and the parents of Phil and Betty work at Pitt and BU, respectively. The internal priority orders based on years of service and preferences of students and colleges are:

Internal Priorities

USC	Pitt	BU
Clay	Phil	Betty
Selma		

Student Preferences

Clay	Selma	Phil	Betty
BU	Pitt	USC	Pitt
$c_\emptyset$	$c_\emptyset$	$c_\emptyset$	$c_\emptyset$

College Preferences

USC	Pitt	BU
Phil	Selma	Selma
Betty	$\emptyset$	$\emptyset$
$\emptyset$		

We read the preferences as follows. Clay prefers BU to being unassigned and prefers being unassigned to other colleges. USC prefers Phil to Betty and prefers both of them to being unassigned.

First, consider the case in which USC decides to export and import only one student. Since Clay is ranked above Selma in its internal ranking (e.g., based on years of service by his father), only Clay will be certified by USC. Suppose that the other two colleges certify their unique students eligible. Clay, Phil, and Betty are the export candidates of USC, Pitt, and BU, respectively. Each export candidate applies to only one college. Only Phil is accepted by the college he is applying to, USC, which runs a positive balance between its imports and exports.

Next, consider the case in which USC decides to export and import two students. Then, additionally, Selma is accepted by the college she is applying to, Pitt. Hence, USC exports one of its students, and the numbers of students it imports and exports are equal.

Now consider the same example with a slight modification of BU's preferences: Suppose BU considers Clay to be the only acceptable student.

In the first case above, Clay and Phil are accepted by BU and USC, respectively, and USC maintains a balance between its imports and exports.

In the second case, additionally, Selma will also be accepted by Pitt. Hence, USC runs a negative balance between imports and exports. $\diamond$

As illustrated in Example 1, it is a very important issue for colleges to decide how many students to certify. If a college certifies too few students, then it will suffer from welfare loss due to both fewer imports and exports. On the other hand, if a college certifies too many students, then it will run a negative imbalance between imports and exports under the current practice, which will harm the school's membership in the program. This feature of tuition exchange process makes it a distinctive allocation problem.

1.4 Model

A **tuition exchange problem** consists of

- a set of **colleges** $C = \{c_1, ..., c_m\}$,
- a set of **students** $S = \underset{c \in C}{\cup} S_c$ where S_c is the set of students who are applying to be sponsored by college c,
- an **import quota** vector $q = (q_c)_{c \in C}$ where q_c is the maximum number of students who will be "imported" by college c,
- an **export eligibility** vector $e = (e_c)_{c \in C}$ where e_c is the number of students in S_c certified eligible by college c,
- a list of college internal rankings $\succ_C = (\succ_c)_{c \in C}$ where $\succ_c$ is the **internal priority order** of students in S_c based on some exogenous rule,
- a list of **student and college preferences** $P = (P_C, P_S) = (\ (P_c)_{c \in C}, (P_s)_{s \in S}\ )$ where P_i is the preference relation of student (college) i over colleges (students) including the remaining unmatched (no more students)

option.[16]

Throughout the paper, C, S, and $\succ_C$ are fixed; each triple of a quota vector, eligibility vector, and a preference profile defines a tuition exchange problem – or simply, a problem – as $[q, e, P]$.

Given a problem, we denote the set of **eligible students** that are certified to be sponsored by college c by E_c where $E_c = \{s \in S_c \mid r_c(s) \leq e_c\}$ where $r_c(s)$ is the rank of student $s \in S_c$ under $\succ_c$. Let $E = \cup_{c \in C} E_c$ be the set of all eligible students.

Each college $c \in C$ has a strict preference relation P_c on $S \cup \emptyset$.[17] We assume that college preferences over admitted groups of students are **responsive** (Roth, 1985) with a quota; that is, for each college c , if P_c^* is the induced preferences over subsets of S by P_c, then for all $T \subset S$ such that $|T| < q_c$ and $i, j \in S \setminus T$, (i) $T \cup i \; R_c^* \; T \cup j \iff i \; R_c \; j$ and (ii) $T \cup i \; R_c^* \; T \iff i \; R_c \; \emptyset$; and for all for all $T \subset S$ such that $|T| > q_c$, $\emptyset P_c^* T$. [18] By a slight abuse of notation, we also refer to the preference relation P_c^* as P_c.

Each student $s \in S$ has a strict preference relation P_s on $C \cup c_\emptyset$. We assume throughout the paper that each student considers her home college *unacceptable*. This assumption is justified by the fact that in tuition exchange

[16]For students, we assume there is an outside option, referring to remaining unmatched within the tuition exchange program, and is denoted by $c_\emptyset$, and for colleges, the no more students option is denoted by $\emptyset$.

[17]Let R_i denote the the at–least–as–good–as relation associated with preference relation P_i for any agent $i \in C \cup S$.

[18]We will denote a singleton $\{x\}$ as x whenever it is convenient.

programs students only rank colleges other than their home colleges. The tuition remission and tuition exchange programs operate separately. [19]

We also assume that there is no tie in the internal priorities. In real life, each college breaks any ties by using lotteries. Hence, each $\succ_c$ is a strict linear order over S_c.

In the tuition exchange problem, students may not consider all colleges worth attending. Also, colleges may not find a student worthy of a scholarship. Hence, we should focus on the matchings that are acceptable for both colleges and students. We say a student s is **acceptable** for college c if $s \; P_c \; \emptyset$ and a college c is **acceptable** by a student s if $c \; P_s \; c_\emptyset$.

An outcome of a problem is a *matching*. A **matching** is a correspondence $\mu : C \cup S \twoheadrightarrow C \cup S \cup c_\emptyset$ such that:

- $\mu(c) \subseteq S$ where $|\mu(c)| \leq q_c$ for all $c \in C$,
- $\mu(s) \subseteq C \cup c_\emptyset$ where $|\mu(s)| = 1$ for all $s \in S$,
- $s \in \mu(c)$ if and only if $\mu(s) = c$ for all $c \in C$ and $s \in S$.

Let $\mathcal{M}$ be the set of matchings. In the tuition exchange problem, only the students who are certified as eligible can be awarded a scholarship. Therefore,

[19] One way to include the option of attending home college as part of the tuition exchange program is to count the students assigned to their home college in both the imported and exported student set of the college. In this case, all our results in this paper go through. As a simplification and to be compatible with the current practice and jargon, we maintain this assumption in this paper.

if student s is not certified as eligible, i.e., if $s \in S \backslash E$, then under any matching $\mu \in \mathcal{M}$ we say that she will be assigned to the null college, $\mu(s) = c_{\emptyset}$.

We introduce several properties of desirable matchings. A matching $\mu \in \mathcal{M}$ **Pareto dominates** another matching $\nu \in \mathcal{M}$ if $\mu(i)\ R_i\ \nu(i)$ for all $i \in C \cup S$ and $\mu(j)\ P_j\ \nu(j)$ for some $j \in C \cup S$. A matching is **Pareto efficient** if it is not Pareto dominated by any other matching.

As we've noted, to stay in good standing in the tuition exchange program, each college must satisfy a balance between its exports and imports. In particular, the number of exports should not exceed the number of imports. Hence, another objective of the colleges is maintaining a one–to–one balance. A matching $\mu \in \mathcal{M}$ is **balanced** if $|X_c^\mu| = |M_c^\mu|$ for all $c \in C$ where $X_c^\mu = \{s \in S_c \mid \mu(s) \in C \backslash c\}$ and $M_c^\mu = \{s \in S \backslash S_c \mid \mu(s) = c\}$ are the **sets of exports** and **imports**, respectively. Note that X_c^μ and M_c^μ are disjoint sets. Balancedness is the key property in a tuition exchange problem. For each tuition exchange problem, the set of balanced matchings is non-empty. For instance, the null matching where all students are unassigned satisfies balancedness. Moreover, there may exist multiple balanced matchings for a given problem. In that case, we can Pareto rank some of the balanced matchings. We say a balanced matching μ is **balanced–efficient** if it is not Pareto dominated by any other balanced matching in which only the certified students are assigned to a college in C. Let b_c^μ be the **net balance** of college c in matching μ and it is equal to $b_c^\mu = |M_c^\mu| - |X_c^\mu|$. We say college c has a *negative* (*positive*) balance in matching μ if $b_c^\mu < 0$ ($b_c^\mu > 0$) and college c has a *zero* balance if

$b_c^\mu = 0$.

A **mechanism** is a systematic way of selecting a matching for each problem. Let φ be a mechanism; then the matching selected by φ in problem $[q, e, P]$ is denoted by $\varphi[q, e, P]$ and the assignment of agent $i \in S \cup C$ is denoted by $\varphi[q, e, P](i)$. A mechanism is **balanced** if it selects a balanced matching in any problem. Similarly, a mechanism is **balanced–efficient** if it selects a balanced–efficient matching in any problem.

In addition to the properties of matchings already defined, we introduce two additional properties for mechanisms. A mechanism ψ is **group strategy-proof for** $N \subseteq C \cup S$ if for all problems $[q, e, P]$, there exists no $J \subseteq N$ and P'_J such that

- $\psi[q, e, (P'_J, P_{-J})](i) \; R_i \; \psi[q, e, P](i)$ for all $i \in J$, and
- $\psi[q, e, (P'_J, P_{-J})](i) \; P_i \; \psi[q, e, P](i)$ for at least one $i \in J$.

Strategy-proofness for a group N is a special case of group strategy-proofness. A mechanism φ is **strategy-proof for** N if it is immune to individual deviations of agents in N, i.e., for $J \subset C \cup S$ such that $|J| = 1$ the above condition holds.[20]

One distinctive feature of the tuition exchange problem is the existence of internal priorities for each college c, $\succ_c$. A good mechanism should respect

[20] Although we formally define incentive compatibility only for preference revelation, we will also inspect incentives in the quota revelation game of colleges.

the internal priorities of each college. It would be desirable that whenever a student s sponsored by college c is allocated to a college in $\varphi[(q_c, q_{-c}), (e_c, e_{-c}), P]$, she should also be assigned in $\varphi[(\tilde{q}_c, q_{-c}), (\tilde{e}_c, e_{-c}), P]$ where $\tilde{e}_c > e_c$ and $\tilde{q}_c \geq q_c$. That is, addition of students with lower internal priority should not cause student s to be unassigned. Formally, a mechanism φ **respects internal priorities** if, whenever a student $i \in S_c$ is assigned to a college in problem $[(q_c, q_{-c}), (e_c, e_{-c}), P]$, then i is also assigned to a college in the problem $[(\tilde{q}_c, q_{-c}), (\tilde{e}_c, e_{-c}), P]$ where $\tilde{e}_c > e_c$ and $\tilde{q}_c \geq q_c$. Given that i is assigned in problem $[(q_c, q_{-c}), (e_c, e_{-c}), P]$, then i's internal rank should be lower than e_c, $r_c(i) \leq e_c$ and all the new students who are sponsored in problem $[(\tilde{q}_c, q_{-c}), (\tilde{e}_c, e_{-c}), P]$ but not in $[(q_c, q_{-c}), (e_c, e_{-c}), P]$ should have higher internal rank (and hence lower internal priority) than i.

1.5 Stability vs. Balancedness: Decentralized vs. Centralized Matching

In the current tuition exchange program, as the centralized process is loosely controlled, once each college sets its export/import quota and eligible students are determined, the market functions more like a decentralized one rather than a centralized one. Once colleges commit to the students they will sponsor, they largely lose their control over them. A sponsored student can sometimes get multiple offers and decide which one to accept and when to accept it. Hence, *stability* emerges as a relevant notion for a benchmark market equilibrium concept when there are no other frictions. Stability has been one of the central solution concepts for two–sided matching problems, and we will also focus on it here for tuition exchange. To define it, we introduce some preliminary concepts. We say a matching μ is **blocked by a college** $c \in C$ if there exists $s \in \mu(c)$ such that $\emptyset \; P_c \; s$. A matching μ is **blocked by a student** $s \in S$ if $c_\emptyset \; P_s \; \mu(c)$. A matching μ is **individually rational** if it is not blocked by any individual college or student. A matching $\mu \in \mathcal{M}$ is **non-wasteful for a pair** $(c, s) \in C \times E$ if, whenever s prefers c to $\mu(s)$ and is acceptable for c, then $|\mu(c)| = q_c$; it is **non-wasteful** if it is non-wasteful for any pair. A matching $\mu \in \mathcal{M}$ is **blocked by a pair** $(c, s) \in C \times E$ if $c \; P_s \; \mu(s)$, and there exists $s' \in \mu(c)$ such that $s \; P_c \; s'$.[21]

A matching μ is **(pairwise) stable** if it is individually rational, non-wasteful, and not blocked by any pair. As we will focus on balanced matchings, we also define a weaker version of stability: A matching μ is **weakly (pairwise) stable** if it is individually rational and not blocked by any pair. Hence, a weakly stable matching can potentially be wasteful. A stable (and hence weakly stable) matching always exists (Gale and Shapley, 1962). A (weakly) stable matching μ is **student optimal (weakly) stable** if there does exist another matching $\nu \in \mathcal{M}$ that is (weakly) stable and Pareto dominates μ.

Below, we characterize weakly stable matchings. Denote the set of all stable matchings and weakly stable matchings for problem $[q, e, P]$ by $\Gamma[q, e, P]$ and $\tilde{\Gamma}[q, e, P]$, respectively. In Proposition 1 we show that the set of weakly

[21]Observe that this definition of blocking is weaker than the traditional definition of blocking, which also incorporates wastefulness for a pair.

stable matchings for a problem is equal to the union of the stable matchings for problems with weakly lower import quotas.

Proposition 1 $\tilde{\Gamma}[q, e, P] = \bigcup_{q' \leq q} \Gamma[q', e, P]$.

We first show that $\tilde{\Gamma}[q, e, P] \subseteq \bigcup_{q' \leq q} \Gamma[q', e, P]$. Let $\mu \in \tilde{\Gamma}[q, e, P]$. We claim that μ is stable for the problem $[q', e, P]$ where $q'_c = |\mu(c)|$ for all $c \in C$. By definition μ is not blocked by any pair $(c, s) \in C \times E$ and is individually rational. Given that all the colleges fill their available seats it is also non-wasteful. Therefore, $\mu \in \Gamma[q', e, P]$.

$\tilde{\Gamma}[q, e, P] \supseteq \bigcup_{q' \leq q} \Gamma[q', e, P]$ follows from the definition of stability and weak stability.

The null matching in which all students are unassigned is a weakly stable matching. Moreover, it is the worst weakly stable matching for the students. In Proposition 2, we show that for any problem, a mutually best weakly stable matching exists for all students.

Proposition 2 *For any problem $[q, e, P]$, the student–optimal weakly stable matching – a weakly stable matching that all students weakly prefer over all other weakly stable matchings – always exists and is the outcome of the student–proposing deferred–acceptance algorithm (DA for short) for problem $[q, e, P]$.*

In Proposition 1 we have shown that the outcome of DA for any problem $[q', e, P]$ is a weakly stable matching where $q' \leq q$. Since DA finds the student–optimal stable matching (cf. Gale and Shapley, 1962), every student likes its outcome for problem $[q', e, P]$ at least as well as any other stable matching for $[q', e, P]$. Due to the resource monotonicity of DA, the matching selected by DA for problem $[q, e, P]$ is weakly preferred to the matching selected by DA for any problem $[q', e, P]$ where $q' \leq q$ by all students (for example, see Kesten, 2006). These together with with Proposition 1 imply that the outcome of DA for problem $[q, e, P]$ is the student–optimal weakly stable matching.

Maintaining a balanced matching is important for the continuation of the membership to the exchange program for a college. If the number of exports of college c exceeds the number of its imports, i.e., if it has a negative balance, then college c will be suspended from TTEI. Hence, balancedness is one of the most important objectives of the colleges. Therefore, the preferences of a college over its incoming class can be linked with the match status of its sponsored students under a particular matching, inducing *college preferences over matchings*. To capture the negative imbalance aversion explicitly, we make the following assumption in some of our results in this section:

Negative–Balance Aversion: College c prefers all μ with $b_c^\mu = 0$ to all ν with $b_c^\nu < 0$, and otherwise, it ranks matchings based on its preferences over the incoming class.

The assumption also makes sure that preferences over matchings are compatible with responsive preferences over the incoming class, which is our main assumption throughout the paper. In particular, under this assumption, our definition of pairwise stability is still valid if we used preferences over matchings instead of preferences over imports for colleges as the primer for measuring the welfare of colleges. This is a reasonable assumption that reflects the real–life preferences of colleges participating in tuition exchange. We will explicitly note when we use this assumption in our results.

We can extend the definition of efficiency and other welfare criteria using college preferences over matchings, as well. In this case, we have the following result:

Proposition 3 *Under negative–balance aversion, if a matching is balanced–efficient then it is also Pareto efficient.*

Suppose μ is a balanced–efficient matching of problem $[q, e, P]$. To the contrary, suppose μ is Pareto dominated. Then there exists an unbalanced matching ν that dominates μ. As it is unbalanced, ν induces a negative balance for some college $c \in C$ (and a positive balance for some other college). However, μ has a zero balance for c, and hence, c prefers μ to ν, which is a contradiction.

We will illustrate how decentralized market forces moving toward stability can be at odds with the college's objective of maintaining a balanced matching. We will also show that colleges always have incentives to decrease their externally set quotas under stable outcomes. Hence, a decentralized market or "stable" centralized mechanisms discourage exchange. In this section, we restrict our attention to the case where each college c sets $e_c = q_c$. Our results do not depend on this assumption, and it is easy to generalize our results for cases where $e_c \neq q_c$ is allowed.

We start with the following proposition, which shows that when the export eligibility and import quotas are set exogenously and each college certifies at least one student as eligible, there may not exist an individually rational and non-wasteful matching that is also balanced.

Proposition 4 *There may not exist a balanced matching that is also individually rational and non-wasteful.*

Consider the following problem. Let $C = \{a, b, c\}$ and each college $d \in C$ sets $q_d = e_d = 1$. The set of students in each college is: $S_a = E_a = \{\mathbf{1}\}$, $S_b = E_b = \{\mathbf{2}\}$ and $S_c = E_c = \{\mathbf{3}\}$. The student preference profile is given as:

1	**2**	**3**
c	a	a
$c_\emptyset$	c	b
	$c_\emptyset$	$c_\emptyset$

Student **1** is not acceptable to college c, i.e., $\emptyset\ P_c\ \mathbf{1}$. Remaining students are acceptable to all colleges (note that we have no restrictions on the strict preferences of colleges over the acceptable students).

There are two non-wasteful and individually rational matchings in this problem:

$$\mu = \begin{pmatrix} a & b & c \\ \mathbf{2} & \mathbf{3} & \emptyset \end{pmatrix} \quad \text{and} \quad \nu = \begin{pmatrix} a & b & c \\ \mathbf{3} & \emptyset & \mathbf{2} \end{pmatrix}.$$

Both matchings fail to be balanced as colleges c and b have negative balances under μ and ν, respectively.

Proposition 4 also shows that there may not exist a stable and balanced matching. However, the problem is not just about non-wastefulness under a preset quota. There may not even exist a weakly stable and balanced–efficient matching as shown in Proposition 5. Recall that a matching is weakly stable if it is individually rational and not blocked by any college–student pair. Weak stability may be relevant when the tuition exchange central office acts cautiously and does not want to admit an additional student to a college without making sure that more of its students are exported.[22]

Proposition 5 *There may not exist a balanced–efficient and weakly stable matching.*

Consider the following problem. There are 4 colleges $C = \{a, b, c, d\}$ and $S_a = E_a = \{\mathbf{1}\}$, $S_b = E_b = \{\mathbf{2}\}$, $S_c = E_c = \{\mathbf{3}\}$, and $S_d = E_d = \{\mathbf{4}\}$. Let $q = (1, 1, 1, 1)$. The preference profiles are:

1	**2**	**3**	**4**	a	b	c	d
b	a	d	a	**4**	**1**	**4**	**3**
$c_\emptyset$	$c_\emptyset$	$c_\emptyset$	c	**2**	$\emptyset$	$\emptyset$	$\emptyset$
			$c_\emptyset$	$\emptyset$			

We will first consider the individually rational and balanced matchings. All individually rational and balanced matchings are:

	a	b	c	d
μ	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
ν	**2**	**1**	$\emptyset$	$\emptyset$
η	$\emptyset$	$\emptyset$	**4**	**3**
π	**2**	**1**	**4**	**3**

[22]A balanced and weakly stable matching always exists; for example, the null matching. However, this is in general not a desired outcome for a tuition exchange problem. A balanced–efficient and individually rational matching also always exists (see the next section).

Balanced matching π Pareto dominates all other balanced matchings; hence it is the unique balanced–efficient and individually rational matching.

Now we check whether π is blocked by a pair or not. In π, $(a, \mathbf{4})$ is a blocking pair because $\mathbf{4}\ P_a\ \pi(a) = \mathbf{2}$ and $a\ P_{\mathbf{4}}\ \pi(\mathbf{4}) = c$; hence, it is not weakly stable.

Proposition 4 shows that there exists no stable and balanced mechanism. One can then wonder whether there exists a stable mechanism that performs better than all the other stable mechanisms in terms of balancedness. In the following proposition we show that for a given tuition exchange problem, in all stable outcomes each college has the same balance. Therefore, the level of imbalance is not affected by which stable mechanism is selected.

Proposition 6 *Each college $c \in C$ has the same balance in all stable matchings of the problem $[q, e, P]$.*

Let ν and μ be any two stable matchings of the problem $[q, e, P]$. In the rural hospital theorem (Roth, 1986) it is shown that the number of students assigned to a hospital is the same in all stable matchings, $|\nu(c)| = |\mu(c)|$ for each college $c \in C$. Moreover, the set of students assigned to a real college is the same in all stable matchings, i.e., $\nu(i) \in C$ if and only if $\mu(i) \in C$. That is, the export set of each college c is the same in all stable matchings. Then, $|M_c^\mu| = |M_c^\nu|$ and $|X_c^\mu| = |X_c^\nu|$. Moreover, because by assumption, no student finds her home college acceptable, she is never assigned to her home college in any stable outcome. Hence, $b_c^\mu = |M_c^\mu| - |X_c^\mu| = |M_c^\nu| - |X_c^\nu| = b_c^\nu$.

Note that if the students are considered acceptable by their home colleges and students rank their home schools as acceptable then Proposition 6 does not hold. Due to the rural hospital theorem, in each stable matching the number of students assigned college $c \in C$ is the same. On the other hand, in some stable matchings the students may be assigned to their home college and they will not be considered as export. Therefore, the number of students exported by each college is not the same in all stable matchings.

As mentioned before, running a negative balance may cause a college to be suspended from the exchange program. Each college prefers to stay in the exchange, and therefore, prefers to prevent any negative balance. Once the colleges report the set of eligible students, they cannot affect the number of exports. Acting cautiously, some colleges set the number of certified eligible students to 1 or 2. However, this may lead to avoidable welfare losses through better coordination. We illustrate this in the following example.

Example 2 Let $C = \{c_1, c_2, ..., c_n\}$ and $|S_c| = k > 1$ for all $c \in C$. Suppose college c_i considers only the students in $S_{c_{i-1}}$ acceptable and each student in $S_{c_{i-1}}$ prefers college c_i to $c_\emptyset$. Let $c_0 = c_n$. Let each college c set $e_c = q_c = 1$. Any stable matching will be balanced and the number of students assigned to colleges will be n. However, if each college c sets $e_c = q_c = k$ then there will be a unique stable matching and it will be balanced. Moreover, the number of students assigned to a college will be $k \times n > n$. $\diamond$

Moreover, if the market is trapped in a low–quota outcome, increasing the quotas incrementally through some minimal coordination among colleges may not immediately lead to a balanced–efficient matching. The following example illustrates this point.

Example 3 Let $C = \{c_1, c_2, ...c_n\}$ and $|S_c| = k > 2$ for all $c \in C$. Suppose:

- For all $i > 1$, the highest priority student of college c_i is the most preferred acceptable student of college c_{i+1} (let $c_{n+1} = c_1$).
- The highest priority student of college c_1 is unacceptable to all colleges.
- For all $i \geq 2$, all other students in S_{c_i} are acceptable to college c_{i+1} and this college is the most preferred college of these students.
- Except the highest priority student of c_1, all students in S_{c_1} are acceptable to all colleges.
- Except the lowest priority student of c_1, all students in S_{c_1} consider c_2 to be unacceptable and the lowest priority student of c_1 prefers c_2 the most.

If each college c sets $q_c = e_c = 1$ then there is a unique stable matching μ with $\mu(c_i) \in S_{c_{i-1}}$ for all $i \neq 2$ and $\mu(c_2) = \emptyset$. The unique stable matching is unbalanced where c_1 has a positive balance and c_2 has a negative balance. Let c_1, the unique college with a positive balance, certify one more student. In this new environment μ is still the unique stable matching. As each college

c with a positive balance increases e_c, we will have a unique stable matching that is unbalanced until we reach $e_1 = k$. $\diamond$

We also investigate what kind of strategic decisions a tuition exchange office in a college would face in quota determination if a stable outcome emerges in the market under the negative–balance aversion assumption. In the following Theorem 1, by using the results of Proposition 7 below, we show that in any stable solution (or mechanism) if a college holds a negative balance then the best response is only to decrease the number of certified eligible students. Proposition 7 also gives us a comparative result regarding how the balances of colleges change when they certify one additional student and do not decrease their import quotas. The proof of the proposition is in Appendix A.3.

Proposition 7 *When college c sets $q_c \geq e_c$ as its import and eligibility quotas, suppose π is a stable matching. When it sets $\tilde{q}_c$ and $\tilde{e}_c$ such that $\tilde{q}_c \geq q_c$ and $\tilde{e}_c = e_c + 1$, suppose $\tilde{\pi}$ is a stable matching. Then $b_c^{\tilde{\pi}} \in \{b_c^{\pi} - 1, b_c^{\pi}\}$ if $b_c^{\pi} < 0$; and $b_c^{\tilde{\pi}} \in \{b_c^{\pi} - 1, b_c^{\pi}, ..., b_c^{\pi} + \tilde{q}_c - q_c\}$ if $b_c^{\pi} \geq 0$.*

The proposition concludes that when a college increases its export eligibility quota by one without decreasing its import quota, its overall balance will decrease at most by one. Its balance may increase only if it is a positive–balance college to start with.[23]

[23] This is possible only if $\tilde{q}_c > q_c$.

Theorem 1 *Under negative–balance aversion, if college c has a negative balance in a stable matching for problem* $[q, e, P]$ *where* $q_c \geq e_c$*, then its best response in any stable solution is to set only lower* q_c *and* e_c*, but not higher; and in particular, there exist* $\tilde{q}_c$ *and* $\tilde{e}_c \leq e_c$ *such that college c has zero–balance in every stable matching of the problem* $[(\tilde{q}_c, q_{-c}), (\tilde{e}_c, e_{-c}), P]$.

Given Proposition 7 and the fact that $\tilde{q}_c = \tilde{e}_c = 0$, we always get a non-negative balance and any college prefers a zero–balance matching to any negative–balance matching.[24]

In Theorem 1, we show that if college c has a negative balance then it tends to decrease the number of certified students and this will eventually increase its balance.[25] When college c certifies fewer students it may cause another college c' to have a negative balance. Then college c' will have a negative balance and will certify fewer students, too. In Theorem 2 below, we show this result. The proof of this theorem is in Appendix A.3.

Theorem 2 *If a college c is holding a negative balance in a stable matching* μ *for problem* $[q, e, P]$ *where* $q_c \geq e_c$ *then* $b^{\mu}_{-c} \geq b^{\mu'}_{-c}$ *where* μ' *is any stable*

[24]Note that setting $\tilde{q}_c = \tilde{e}_c = 0$ may not be the only way to have a zero balance.

[25]This result is in a similar vein as the results on college admissions where the DA mechanism is shown to be prone to import quota manipulation of the colleges under responsive preferences, regardless of imbalance aversion (cf. Sönmez, 1997). However, Konishi and Ünver (2006) show that the DA mechanism would be immune to quota manipulation, if preferences of colleges over incoming students were responsive and monotonic in number. On the other hand, even under this restriction of preferences over the incoming class, our result would imply all stable mechanisms are manipulable with quota reports for colleges with negative balances if colleges have preferences with negative–balance aversion over matchings. (See also **?**.)

matching of problem $[(q'_c, q_{-c}), (e'_c, e_{-c}), P]$, $q_c \geq q'_c \geq e'_c$ *and* $e_c > e'_c$.

Theorems 1 and 2 do not constitute an equilibrium analysis in a quota determination game. But they do point out that in a frictionless market, the negative–balance colleges will be conservative and decrease their eligibility quotas for exports, certifying fewer students, which will further deteriorate the balances of colleges who had negative balances to start with. On the other hand, in Appendix A.5, we analyze the equilibrium in a quota determination game when colleges are positive–balance–averse. Positive–balance aversion can be justified in markets where transactions do not need to balance out; colleges with a positive balance admit more students than they need to, which is not desirable. Tuition exchange programs such as The Council of Independent Colleges Tuition exchange program (CIC-TEP) do not require balanced exchanges. In such markets, colleges may prefer not to admit more students than they export.

We illustrate Theorem 2 with the following example.

Example 4 There are n colleges, $C = \{c_1, c_2, ..., c_n\}$ and each college has m students. Consider the following problem:

- Each college sets $e_c = q_c = m$.
- Each student in S_{c_k} prefers c_{k+1} most where $c_{n+1} = c_1$. Each student in $S \setminus S_{c_{n-1}}$ prefers being unassigned to any other college. Each student in $S_{c_{n-1}}$ ranks c_1 second and being unassigned third.

- Each student in $S \setminus S_{c_n}$ is acceptable to all colleges. All students in S_{c_n} are unacceptable to c_1.

There exists a unique stable matching for the given problem. In this stable matching only c_1 has a negative balance. If c_1 certifies $m-1$ (and sets $q_{c_1} = m-1$) students, then there exists a unique stable matching in which only c_1 and c_2 have a negative balance. If c_1 and c_2 certify $m-1$ students, then in the unique stable matching only c_1 and c_3 have a negative balance. If c_1, c_2 and c_3 certify $m-1$ students, then in the unique stable matching only c_1 and c_4 have a negative balance. Finally, we will have a problem where all colleges except c_n certify $m-1$ students and only c_1 has a negative balance. If we continue we can get a matching problem in which all colleges in $C \setminus c_n$ certify zero students and there exists a unique stable matching: the null matching.

On the other hand there exists a unique balanced–efficient matching in which all students in $S \setminus S_n$ are assigned to their top or second best choices. $\diamond$

However we can show that there exists a balanced–efficient mechanism under which it is a dominant strategy to certify the maximum number of students that each college is willing to sponsor.

We conclude that under a new design for the tuition exchange market, there should be no need for external quota determination by the colleges due to negative–balance aversion. A fully centralized solution disregarding stability seems to be inevitable, as stability is at odds with balancedness and has various other shortcomings regarding other incentives. In our proposed design, we will stick to balanced–efficiency and individual rationality as our desired features. It turns out that we can find a plausible balanced–efficient mechanism and individually rational mechanisms that make it a dominant strategy for a college to certify its full set of eligible students (see Proposition 5 below). Under a centralized mechanism, incentives for participants to truthfully reveal their preferences are desirable. Unfortunately, we show that balanced–efficiency, individual rationality, and strategy-proofness for colleges are incompatible properties:

Proposition 8 *There does not exist an individually rational and balanced–efficient mechanism that is also incentive compatible for colleges.*

Suppose there does exist such a mechanism. Denote it by ψ. In this proof we will use different examples to show our result.

Case 1: There are 3 colleges $C = \{a, b, c\}$ and 4 students $S_a = \{1, 2\}$, $S_b = \{3\}$ and $S_c = \{4\}$. Let $q = e = (2, 1, 1)$. Each student considers all colleges acceptable. The preferences of colleges are given as: $P_a : 3P_a4P_a\emptyset$, $P_b : 1P_b\emptyset$ and $P_c : 1P_c2P_c\emptyset$. There are 2 balanced–efficient and individually rational matchings:

	a	b	c
μ_1	$\mathbf{4}$	$\emptyset$	$\mathbf{1}$
μ_2	$\mathbf{3, 4}$	$\mathbf{1}$	$\mathbf{2}$

If the outcome of ψ is μ_1, then college a can manipulate the mechanism by submitting $P'_a : 3P'_a\emptyset$. Then the only individually rational and balanced–

efficient matching is $\mu_3(a) = 3$, $\mu_3(b) = 1$ and $\mu_3(c) = \emptyset$. Therefore, if there is a mechanism that is incentive compatible for colleges, balanced–efficient, and individually rational, then it selects μ_2.

Case 2: We consider the same example with a slight change in college a's preferences: $P_a : 4P_a3P_a\emptyset$. In this case μ_1 and μ_2 are the only two balanced–efficient and individually rational matchings.

If the outcome of ψ is μ_1, then college a can manipulate the mechanism by submitting $P'_a : 3P'_a4P'_a\emptyset$. Then we will be in case 1, in which case we show that μ_2 will be selected and this makes a better off. Therefore, if there is a mechanism that is incentive compatible for colleges, balanced–efficient, and individually rational, then it selects μ_2.

Case 3: Now consider the case where colleges report the following preferences: $P_a : 4P_a3P_a\emptyset$, $P_b : 1P_b\emptyset$, and $P_c : 1P_c\emptyset$. Then, there are two individually rational and balanced–efficient matchings:

	a	b	c
μ_4	**4**	$\emptyset$	**1**
μ_5	**3**	**1**	$\emptyset$

If the outcome of ψ is μ_4, then in case 2 college c can violate the mechanism by excluding 2 from its preferences. Therefore, any balanced–efficient and individually rational mechanism that is incentive compatible for colleges selects μ_5.

Case 4: Now consider the case where colleges report the following preferences: $P_a : 4P_a\emptyset$, $P_b : 1P_b\emptyset$ and $P_c : 1P_c\emptyset$. There is a unique balanced-efficient and individually rational matching: μ_4. Note that in case 3 college a can manipulate the mechanism by excluding 3 from its preferences; then we will be in case 4 and a will be better-off.

Therefore, there does not exist a strategy-proof mechanism that is also balanced–efficient and individually rational.

Although there doesn't exist a balanced–efficient and individually rational mechanism that cannot be manipulated by colleges, in the next section we show that there exists a balanced–efficient and individually rational mechanism that cannot be manipulated by students. Moreover, colleges can manipulate our proposed mechanism by reporting an acceptable student as unacceptable; but they cannot manipulate it either through export eligibility/import quota manipulation or by ranking acceptable students untruthfully.

1.6 A Balanced–Efficient Mechanism: Two–Sided Top Trading Cycles

In this section, we consider the model in which all colleges are restricted to maintain a zero balance in every period. We propose a mechanism that is individually rational, balanced–efficient, and strategy-proof for students. Moreover, it respects the internal priority orders of the colleges, i.e., it respects internal priorities. In Appendix 15, we relax the zero balance requirement and allow colleges to have a balance in a predetermined interval. This relaxation also allows us to study the problem in a dynamic environment where a cumulative balance over the years determines the continuation of membership.

As mentioned in the previous section, the assignment selected for each problem should be balanced; a college with a negative balance can be suspended from the program. If a matching is balanced then the number of students exported by each college c is equal to the number of students imported by that college. We can consider a balanced matching as a one–to–one trade between each college.

Before introducing a new mechanism that satisfies the desired features, we define a cycle. A **trading cycle** is an ordered list of colleges and students $(c_1, s_1, c_2, s_2, ..., c_k, s_k)$ such that:

- college c_1 points to student s_1,
- student s_1 points to college c_2,
- $\vdots$ $\vdots$
- college c_k points to student s_k,
- student s_k points to college c_1.

A **trivial cycle** is a college–student pair (c, s) such that college c points to student s and student s points to college c.

In the following proposition, we show that if a matching is balanced then we can find a finite number of trading cycles involving colleges and students who do not remain unmatched.

Proposition 9 *A matching μ is balanced if and only if each student s assigned to a college is in a unique cycle where she points to $\mu(s)$ and is pointed to by her home college c.*

We propose a variant of the *top–trading–cycles mechanism* (TTC). Variants of TTC have been studied in matching literature for one–sided discrete resource allocation problems such as school choice (Abdulkadiroğlu and Sönmez, 2003) and dormitory room allocation at college campuses (Abdulkadiroğlu and Sönmez, 1999). TTC was based on *Gale's top–trading–cycles algorithm* (Shapley and Scarf, 1974), which was used to find the core allocation of a simple discrete exchange economy, commonly referred to as the *housing market*. In each of these problems, there is an active side of agents who are exchanging objects that are allocated either through individual property rights or through the mechanism's definition to agents as endowments (also see Pápai, 2000; Pycia and Ünver, 2011a). However, in our problem college slots are not objects, as the colleges are active decision makers. To capture this difference, we propose a two–sided version of the TTC. For any given problem $[q, e, P]$, it works iteratively in a number of rounds:

The Two–Sided Top–Trading–Cycles Mechanism (2S-TTC):

Round 1: Assign two counters, for import and export eligibility, for each college $c \in C$, and set them equal to q_c and e_c, respectively. Each student points to her favorite college in $C \cup c_\emptyset$ that considers her acceptable, and each college $c \in C$ points to the student $s \in S_c$ who has the highest internal priority. The null college $c_\emptyset$ points to the students pointing to it, i.e., the students pointing to the null college. Because of the finiteness of colleges and students, there exists at least one cycle. Each (real) college and student can be part of at most one cycle. Every student in the cycle is assigned a seat at the college she points to and is removed. If the cycle is nontrivial then the counters of each college in that cycle are reduced by one, and if any of them reaches zero, the college is removed with its remaining students. If the cycle is trivial then we reduce only the export counter of the college whose student is in that cycle, and if it reaches zero, the college is removed.

In general, at

Round k: Each remaining student points to her favorite college in $C \cup c_\emptyset$ that considers her acceptable among the remaining ones, and each remaining college c points to the student $s \in S_c$ who has the highest internal priority among the remaining ones. The null college $c_\emptyset$ points to the students pointing to it, i.e., the students pointing to the null college. There exists at least one cycle. Each (real) college and student can be part of at most one cycle. Every student in the cycle is assigned a seat at the college she points to and is removed. If the cycle is nontrivial then the counters of each college in that cycle are reduced by one, and if any of them reaches zero, the college is removed with its remaining students. If the cycle is trivial then we reduce only the export counter of the college whose student is in that cycle, and if it reaches zero, the college is removed.

The algorithm terminates when there are no remaining eligible students in the problem. We illustrate the dynamics of the 2S-TTC mechanism with an example below:

Example 5 (**2S-TTC**) Let $C = \{a, b, c, d, e\}$, $S_a = \{\mathbf{1}, \mathbf{2}\}$, $S_b = \{\mathbf{3}, \mathbf{4}\}$, $S_c = \{\mathbf{5}, \mathbf{6}\}$, $S_d = \{\mathbf{7}, \mathbf{8}\}$, and $S_e = \{\mathbf{9}\}$. Let each college certify all its students as eligible and $q = (2, 2, 2, 1, 1)$. The internal priority orders are given as:

a	b	c	d	e
1	**3**	**6**	**7**	**9**
2	**4**	**5**	**8**	

The preference profiles of colleges and students are given as:

a	b	c	d	e
3	**5**	**2**	**2**	**2**
4	**1**	**3**	**3**	**3**
5	**6**	**4**	**4**	**8**
9	**2**	**9**	**9**	**7**
7	**7**	**7**	**5**	**5**
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
b	b	a	c	b	a	c	e	c
c	c	c	a	a	b	a	c	d
$c_\emptyset$	$c_\emptyset$	$c_\emptyset$	$c_\emptyset$	$c_\emptyset$	$c_\emptyset$	$c_\emptyset$	$c_\emptyset$	$c_\emptyset$

Let o_e and o_m be the vectors representing the export eligibility and import counters of colleges, respectively. Then we set $o_e = (2, 2, 2, 2, 1)$ and $o_m = (2, 2, 2, 1, 1)$.

Round 1: The only cycle formed is $(b, \mathbf{3}, a, \mathbf{1})$. Therefore, **1** is assigned to b and **3** is assigned to a. Observe that although college b is the most preferred

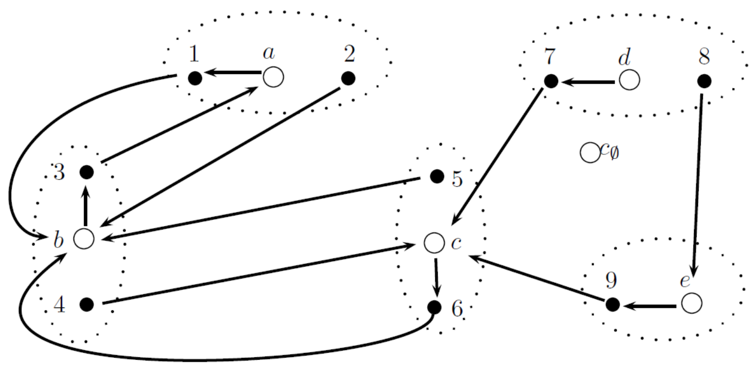

Figure 1.1: Round 1 of Example 5

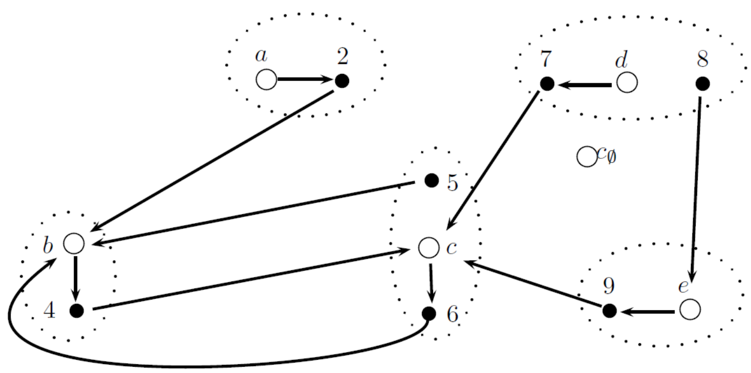

Figure 1.2: Round 2 of Example 5

college of student **6**, she is not acceptable to b, and hence, she points to college a, instead. The updated counters are $o_e = (1,1,2,2,1)$ and $o_m = (1,1,2,1,1)$.

Round 2: The only cycle formed in Round 2 is $(c, \mathbf{6}, b, \mathbf{4})$. Therefore, **6** is assigned to b and **4** is assigned to c. The updated counters are $o_e = (1,0,1,2,1)$ and $o_m = (1,0,1,1,1)$. College b is removed.

Round 3: The only cycle formed in Round 3 is $(a, \mathbf{2}, c, \mathbf{5})$. Therefore,

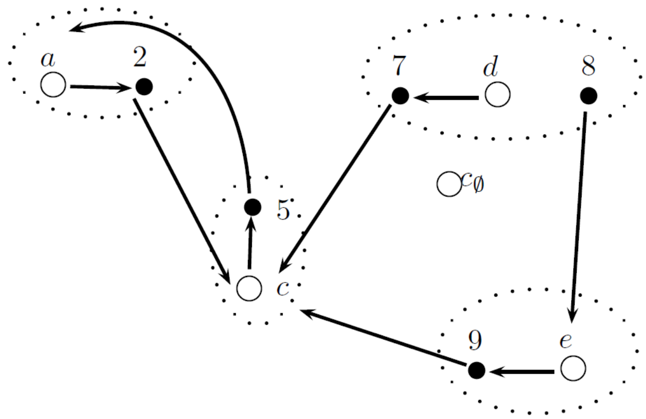

Figure 1.3: Round 3 of Example 5

5 is assigned to a and **2** is assigned to c. The updated counters are $o_e = (0, 0, 0, 2, 1)$ and $o_m = (0, 0, 0, 1, 1)$. Colleges a and c are removed.

Round 4: The only cycle formed in Round 4 is $(c_\emptyset, \mathbf{7})$. Therefore, **7** is assigned to $c_\emptyset$. Given that we have a trivial cycle, we only update o_e. The updated counters are $o_e = (0, 0, 0, 1, 1)$ and $o_m = (0, 0, 0, 1, 1)$.

Round 5: The only cycle formed at this round is $(e, \mathbf{9}, d, \mathbf{8})$. Therefore, **8** is assigned to e and **9** is assigned to d. The updated counters are $o_e = (0, 0, 0, 0, 0)$ and $o_m = (0, 0, 0, 0, 0)$. All agents are assigned, so the algorithm terminates and its outcome is given by matching

$$\mu = \begin{pmatrix} a & b & c & d \\ \{\mathbf{3}, \mathbf{5}\} & \{\mathbf{1}, \mathbf{6}\} & \{\mathbf{2}, \mathbf{4}\} & \mathbf{9} \end{pmatrix}.$$

◇

In the following theorem, we show that 2S-TTC is balanced–efficient

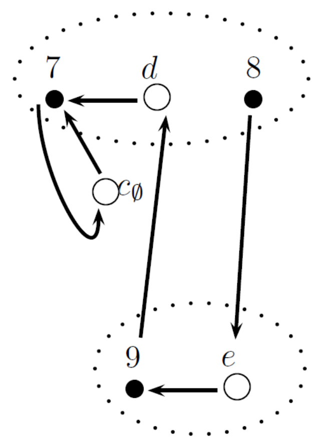

Figure 1.4: Round 4 of Example 5

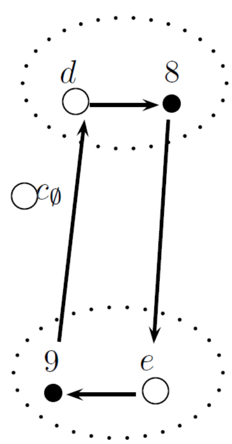

Figure 1.5: Round 5 of Example 5

and individually rational, and respects internal priorities. We prove this in Appendix A.3.

Theorem 3 *2S-TTC is a balanced–efficient and individually rational mechanism that also respects internal priorities.*

In Proposition 8, we show that there does not exist an individually rational and balanced–efficient mechanism that is also strategy-proof for colleges. From Proposition 8 and Theorem 3, it is easy to see that the 2S-TTC mechanism is not strategy-proof for colleges.

Proposition 10 *2S-TTC is not strategy-proof for colleges.*

It follows from Proposition 8 and Theorem 3.

Although 2S-TTC is not strategy-proof, in the following theorem we show that it is group strategy-proof for students. This result is a consequence of TTC being group strategy-proof in a house allocation/exchange market (cf. Pápai, 2000).

Theorem 4 *2S-TTC is group strategy-proof for students.*

Consider the preference relations of each student that rank as acceptable only the colleges that find her acceptable. If we consider only these preferences as possible preferences to choose from for each student, we see that 2S-TTC is group strategy-proof for students, as Pápai (cf. 2000) showed that TTC is group strategy-proof. In 2S-TTC, observe that students are indifferent among reporting preference relations that rank the colleges that find themselves as acceptable in the same relative order. Thus, 2S-TTC is group strategy-proof for students.

In Proposition 10 we show that 2S-TTC mechanism is not strategy-proof for colleges. However, if we focus on the game played by the tuition-exchange office of a college, given its preferences, truthful import quota revelation and certifying all its own students is a (weakly) dominant strategy under 2S-TTC. The proof is in Appendix A.3.

Theorem 5 *It is a weakly dominant strategy for any $c \in C$ to certify all its students in 2S-TTC and to reveal its true import quota under any fixed reported preference profile such that c does not report an unacceptable student as acceptable in its preference report.*

Moreover, reporting an unacceptable student as acceptable in its preference report is a dominated strategy in the mechanism game.

Theorem 5 is in stark contrast with similar results in the literature for stable mechanisms. For example, it is well known that the DA mechanism is prone to import quota manipulation of the colleges even under responsive preferences regardless of imbalance aversion (cf. Sönmez, 1997); truthful revelation does not even constitute an equilibrium, and any pure strategy equilibrium, if it exists, increases all colleges' welfare above truthful revelation (cf. Konishi and Ünver, 2006). Thus, 2S-TTC presents a robust remedy for a common problem seen in centralized admissions that use the DA mechanism (such as K-12 public school choice in US school districts) and also in tuition exchange in a decentralized market (cf. Theorems 1 and 2).

Theorem 4 and Proposition 5 point out that only colleges can benefit from manipulation, and they can manipulate the 2S-TTC mechanism by misreporting their preferences over incoming students. However, as we only care whether a college finds a student acceptable or not in running 2S-TTC, it can be run as an indirect mechanism where colleges report only their acceptable incoming students. Hence, the strategy space for the colleges is very simple in using 2S-TTC in the field: reporting their import and eligibility quotas and their sets of acceptable students.

Corollary 1 *Under the 2S-TTC mechanism, colleges are indifferent among strategies with preference rankings considering the same set of students as acceptable.*

In the market, we propose to run 2S-TTC in sequential stages in a semi-decentralized fashion: first, colleges announce their import quotas and which of their students are eligible to be sponsored; then, eligible students apply to the colleges they find acceptable; colleges send out admission letters. At this stage as students have also learnt their opportunities in the parallel running regular college admissions market, they can form better opinions about the relative ranking of the null college, i.e., their options outside the tuition exchange market. Students submit ranksings over the colleges that admitted them and the relative ranking of their outside option. Finally, 2S-TTC is run centrally to determine the scholarship recipients.

We have shown that 2S-TTC has appealing properties. In the following theorem, we show that it is the unique mechanism satisfying respect for internal priorities, individual rationality, balanced–efficiency, and strategy-proofness for students. We prove this in Appendix A.3.

Theorem 6 *2S-TTC is the unique student–strategy-proof, individually rational, and balanced–efficient mechanism that also respects internal priorities.*

Below we show the independence of axioms mentioned in Theorem 6.

- *A student–strategy-proof, individually rational but not balanced–efficient mechanism that also respects internal priorities:* A mechanism that always selects the null matching for any problem.

- *A student–strategy-proof, balanced–efficient, individually rational mechanism that does not respect internal priorities:* Consider a variant of 2S-TTC mechanism in which each college points to the certified student who has the lowest priority among the certified ones. This mechanism is strategy-proof for students, balanced–efficient, and individually rational, but it fails to respect internal priorities.

- *A balanced–efficient, individually rational, but not student–strategy-proof mechanism that respects internal priorities:* Consider the following problem. There are three colleges $C = \{a, b, c\}$ and four students $S_a = \{\mathbf{1}, \mathbf{2}\}$,

$S_b = \{\mathbf{3}\}$ and $S_c = \{\mathbf{4}\}$. The preference profile P is given as

1	**2**	**3**	**4**
b	b	a	a
c	c	$c_\emptyset$	$c_\emptyset$
$c_\emptyset$	$c_\emptyset$		

Let mechanism ψ select the same matching as 2S-TTC for each problem except the problem $[q = (2,1,1), e = (2,1,1), P]$, and for this problem it assigns **1** to c **2** to b, **3** to a and **4** to a. The mechanism is not student–strategy-proof, because when **1** excludes c, ψ and 2S-TTC will assign **1** to b.

- *A balanced–efficient, student–strategy-proof, but not individually rational mechanism that respects internal priorities:* Consider a variant of 2S-TTC mechanism in which students are not restricted to point to the colleges considering them acceptable. This mechanism is balanced–efficient, strategy-proof, and respecting internal priorities, but it fails to be individually rational since an unacceptable student can be assigned to a college.

We show that 2S-TTC is the unique mechanism satisfying all the axioms discussed. Among all axioms, only the respect for internal priorities is based on exogenous rules. Then, one might suspect that more students will benefit from the tuition exchange program if we allow the violation of respect for internal priorities. A natural question that arises is whether there is a balanced and individually rational mechanism that never assigns fewer students than the 2S-TTC mechanism and selects a matching in which more students are assigned whenever there exists such an outcome. In the following proposition, we show that the mechanism satisfying the above conditions is vulnerable to manipulation.

Proposition 11 *Any balanced and individually rational mechanism that does not assign fewer students than the 2S-TTC mechanism, and selects a matching in which more students are assigned whenever such a balanced and individually rational outcome exists, is not strategy-proof for students.*

Let ψ satisfy all conditions and be strategy-proof for students. Then, consider the following example. There are 3 colleges $C = \{a, b, c\}$ with $q = (2, 1, 1)$. Let $S_a = \{\mathbf{1}, \mathbf{2}\}$, $S_b = \{\mathbf{3}\}$, $S_c = \{\mathbf{4}\}$ and let each student be acceptable to each college. The internal priorities and preference profiles are given as:

a	b	c
1	**3**	**4**
2		

1	**2**	**3**	**4**
b	c	a	b
$c_\emptyset$	$c_\emptyset$	$c_\emptyset$	a
			$c_\emptyset$

The 2S-TTC mechanism will select matching $\mu = \begin{pmatrix} a & b & c \\ \{\mathbf{3}, \mathbf{4}\} & \mathbf{1} & \mathbf{2} \end{pmatrix}$. Moreover, ψ will select the same matching.

If student **4** reports her preferences as $bP'_{\mathbf{4}}c_\emptyset P'_{\mathbf{4}}a$ then 2S-TTC will select $\mu' = \begin{pmatrix} a & b & c \\ \mathbf{3} & \mathbf{1} & \emptyset \end{pmatrix}$. The only balanced and individually rational matching in which more than two students are assigned is $\mu'' = \begin{pmatrix} a & b & c \\ \mathbf{3} & \mathbf{4} & \mathbf{2} \end{pmatrix}$. Therefore, the outcome of mechanism ψ when **4** reports $P'_{\mathbf{4}}$ is μ''. Student **4** can manipulate mechanism ψ since she strictly prefers her assignment under misreporting.

1.7 Stability and Repeated Deferred–Acceptance Mechanism

In this section, we focus on the tuition exchange co-ops where stability is valued over balancedness.[26] In these programs, colleges are not punished for holding a negative balance, but holding a zero balance is still a desired feature and plays key role in the longevity of the programs.[27] In this section, we aim to come up with a stable mechanism that lowers the imbalance as far as possible.

For a tuition exchange program that favors stability over balancedness, the following mechanism can be considered a good option.[28]

The Repeated Deferred–Acceptance Mechanism (RDA) :

Let $\tilde{e}_c = \min\{q_c, e_c\}$ for each school c.

Round 1: Run the DA mechanism for problem $[q, \tilde{e}, P]$. Denote the matching selected in this round by μ_1. If there exists a college c with a positive

[26]We provide details about some of these programs in Appendix A.1.

[27]For example, the Northwest Independent Colleges Tuition Exchange Program, which does not require any sort of balancedness criterion, will be dissolved in 2015 because of accumulation of imbalances (cf. Basu, 2012).

[28]For completeness of the definition of our mechanism, we remind the reader of the student–proposing deferred–acceptance algorithm of Gale and Shapley (1962) in Appendix A.3. We refer to the mechanism that finds an outcome according to this algorithm simply as the DA mechanism.

balance, $b_c^{\mu_1} > 0$ and $\tilde{e}_c < e_c$, then certify a new student in S_c by increasing $\tilde{e}_c$ by 1. Keep the export eligibility quotas of other colleges unchanged. If there is more than 1 college with a positive balance, take one of them randomly. If there does not exist a college with a positive balance or $\tilde{e}_c = e_c$ for all colleges c with positive balances, then stop.

In general, at

Round k: Run the DA mechanism for the updated problem $[q, \tilde{e}, P]$. Denote the matching selected in this round by μ_k. If there exists a college c with a positive balance, $b_c^{\mu_k} > 0$ and $\tilde{e}_c < e_c$, then certify a new student in S_c by increasing $\tilde{e}_c$ by 1. Keep the export eligibility quotas of other colleges unchanged. If there is more than 1 college with a positive balance, take one of them randomly. If there does not exist a college with a positive balance or $\tilde{e}_c = e_c$ for all colleges c with positive balances, then stop.

The mechanism will terminate, since each college has a finite number of students to sponsor.

If the mechanism terminates since there does not exist a college with a positive balance, then the matching selected in that round is balanced.

However, recall that it is not guaranteed that we can get a balanced and stable matching for all problems (Proposition 4).

In the following proposition, we show that, while running the RDA mechanism, the balance of each college c whose export eligibility quota is the same in any Rounds k and k+1 will stay the same or increase between these rounds.

Proposition 12 *If the export eligibility quota of college c is the same in Rounds k and k+1 of RDA, then* $b_c^{\mu_{k+1}} \in \{b_c^{\mu_k}, b_c^{\mu_k}+1\}$*, and otherwise,* $b_c^{\mu_{k+1}} \in \{b_c^{\mu_k}-1, b_c^{\mu_k}\}$ *and* $0 \leq b_c^{\mu_{k+1}}$*.*

It follows from Lemma 1 stated in Appendix A.3.

As can be seen from Proposition 12, only one college's balance deteriorates, and that college has a positive balance in the previous period. Since the deterioration will be at most by 1, that college will still have a non-negative balance in the outcome selected in the next round. On the other hand, all other colleges' balances will weakly improve. Therefore, if college c has a positive balance in the outcome selected in round k, $b_c^{\mu_k} > 0$, then independent of the order of the colleges selected to certify new students in the following rounds, college c will certify at least $\min\{b_c^{\mu_k}, e_c - \tilde{e}_c\}$ students before the RDA mechanism terminates.

In the following proposition we show that the outcome of the RDA mechanism is independent of the order of student certification. Its proof is in Appendix A.3.

Proposition 13 *The outcome of RDA is independent of which colleges we choose to update in each round.*

Then consider the following selection rule: Let $\tilde{S} = \{s \in S_c | b_c^{\mu_k} > 0,\ \tilde{e}_c < r_c(s) \leq \min\{\tilde{e}_c + b_c^{\mu_k}, e_c\}\}$. That is, $\tilde{S}$ is the set of students who are guaranteed to be certified as a consequence of the outcome selected in round 1. Then, in the following $|\tilde{S}|$ rounds, we always certify a student from $\tilde{S}$. Based on this selection rule in round $|\tilde{S}| + 1$ the set of certified students will be the union of the initial certified students and $\tilde{S}$. Instead of certifying one student in each round, we can certify all students in $\tilde{S}$ at the end of round 1 and jump to round $|\tilde{S}| + 1$. That is, we can update the export eligibility quota of each college with a positive balance at the same time, and this will decrease the number of times that we need to run the DA mechanism.

Now we can look at which other appealing properties are satisfied by RDA. We prove this theorem in Appendix A.3.

Theorem 7 *RDA is strategy-proof for students.*

In order to compare the performance of weakly stable matchings and the outcome of RDA in terms of balancedness, we introduce a new measure. Let B^μ be the **aggregate balance** of matching μ and $B^\mu = \sum_{c \in C} |b_c^\mu|$. In the following proposition, we show that if we were to introduce more students above the eligible set in the round that the RDA mechanism terminates, any stable outcome of that market will have a worse aggregate balance. That is, RDA optimizes aggregate balance. The proof is in Appendix A.3.

Proposition 14 *Let μ be the matching selected by RDA for problem $[q, e, P]$ and $\tilde{e} = (\tilde{e}_c)_{c \in C}$ be the vector of certified students when RDA terminates. Let*

ν be a stable matching for a problem $[q, e', P]$ where $e_c \geq e'_c \geq \tilde{e}_c$ for all $c \in C$. Then $B^\mu \leq B^\nu$.

Corollary 2 *Let $\tilde{e}$ be the vector of the numbers of certified students when RDA terminates in problem $[q, e, P]$. If the final matching selected by RDA is not balanced then there does not exist a stable and balanced matching for any problem $[q, e', P]$ where $e_c \geq e'_c \geq \tilde{e}_c$ for all $c \in C$.*

Corollary 3 *Let $\underline{e} = (\min\{q_c, e_c\})_{c \in C}$ be the number of certified students that we consider in the first round of RDA and $e'_c \geq \underline{e}_c$ for all $c \in C$. If there exists a balanced and stable matching for the problem $[q, e', P]$ then the outcome of RDA is also balanced.*

The RDA mechanism can be proposed to the Council of Independent College Tuition Exchange Program (CIC-TEP) (see Appendix A.1 for details). Although CIC-TEP does not require balancedness, it is widely agreed that a method to balance the benefit and cost between schools is a key goal of all tuition exchange programs. Otherwise, some institutions might carry an inordinate number of consortium dependent children. In the current system, all colleges are required to import at least 3 students. There is no feature that limits the monetary imbalance of higher–tuition institutions exchanging students with lower–tuition institutions. This may result in the membership composition shifting to less prestigious colleges with lower annual tuition costs. In practice, we can fix each college's quota to 3 and apply RDA. Note that in the current practice there is no limit on exports, and hence each student is considered to be eligible and sponsored by her home college.

1.8 College Preferences Over Outgoing Students and TTC

In earlier sections, we defined the preferences of colleges over incoming students, and avoided explicitly modeling their preferences over the outgoing class other than noting in a few instances that colleges would like to avoid a negative balance. Modeling colleges with preferences over incoming students is consistent with the rest of the literature. In contrast with the markets studied earlier, in a tuition exchange market, each college has outgoing students. Since a tuition exchange program is considered a benefit for faculty members, college preferences can be taken over the outgoing students, as well. In this section, we consider the case in which colleges have preferences over outgoing students and are indifferent among acceptable incoming students.

If balancedness is required and colleges are assumed to care only about their exports, then the tuition exchange office of the college would like to accept all of its applicants, including the unacceptable ones. In real life, an unacceptable student, a student who is not admitted during the regular admission procedure, cannot be awarded a scholarship. We will use two assumptions about college preferences over matchings in some of our results in this section; otherwise no explicit assumptions are made about college preferences.

Responsive Preferences Over Export Students: For each college c, if $\succ_c^*$

is the induced preferences over subsets of S_c by internal priority order of college c, $\succ_c$, then for any $T \subset S_c$ with $|T| < e_c$ and $i,\ j \in S_c \setminus T$, (i) $T \cup i \succ_c^* T \cup j \iff i \succ_c j$ and (ii) $T \cup i \succ_c^* T \iff r_c(i) \leq e_c$. College c's preferences over matchings are determined with respect to its preferences over export students.

No Unacceptable Import Students: College c prefers a matching where it imports and exports no students to any matching with an unacceptable import student.

Recall that colleges have three components in their "strategy" to play with in a tuition exchange market: (1) revealed preferences over applying import students, (2) import quota, and (3) export eligibility quota. Here, we assume that colleges are indifferent among incoming students as long as they are acceptable; hence the above interpretation of strategies is consistent also for direct mechanisms in this setting. The internal priority order is exogenously given by the bylaws of the university, which cannot be easily changed or manipulated as they represent a promise to the hired faculty members. However, colleges can and still do affect the set of students that they will export by acting strategically when they report import and export eligibility quotas (see Theorem 8 below). In spite of these real-life interpretations, we should note that our results are robust to the case where colleges are assumed to determine the internal priority order strategically.

As in the previous sections, our main focus is finding a mechanism that satisfies strategy-proofness, individual rationality, and balanced–efficiency. In

Theorem 8 we show that there does not exist a mechanism satisfying the previously defined desirable properties. The proof is in Appendix A.3.

Theorem 8 *Under assumptions of responsive preferences over export students and no unacceptable import students, there does not exist a balanced–efficient and individually rational mechanism that is immune to import preference and quota manipulation.*

In Section 1.6, we show that the 2S-TTC mechanism is the unique mechanism satisfying individual rationality, balanced–efficiency, respect for internal priorities, and strategy-proofness for students. It is easy to see that, 2S-TTC satisfies strategy-proofness for students and respects internal priorities independent of the assumption about college preferences. As long as colleges report their acceptable set, 2S-TTC will still be individually rational when we define colleges' preferences over the outgoing students. Under the assumption of no unacceptable import students, any matching in which an unacceptable student is assigned to a college cannot Pareto dominate the outcome of the 2S-TTC mechanism. When we prove that the outcome of the 2S-TTC mechanism cannot be Pareto dominated by a balanced and individually rational matching, we consider the preferences of the students and we only look at whether the student is acceptable or not. This is also true in the uniqueness proof. Therefore, changing the assumption about the preferences of colleges does not change our result. We state our uniqueness result under the assumption of no unacceptable import students in Theorem 9. We prove it in Appendix A.3.

Theorem 9 *Under the assumption of no unacceptable import students, the 2S-TTC mechanism is the unique mechanism satisfying individual rationality, balanced–efficiency, respect for internal priorities, and strategy-proofness for students.*

In Theorem 8 we showed that there is no individually rational and balanced–efficient mechanism that is also strategy-proof when colleges have preferences over outgoing students. Given that the 2S-TTC mechanism is balanced–efficient, individually rational, and strategy-proof for students, it should fail to be strategy-proof for colleges. That is, colleges benefit either from quota manipulation or from misreporting preferences over the incoming students or both. In Theorem 10 we show that colleges can only benefit from misreporting their preferences over the incoming students. We prove it in Appendix A.3.

Theorem 10 *When preferences of colleges are responsive over export students, the 2S-TTC mechanism is immune to quota manipulation.*

1.9 Conclusion

In this paper, we introduce a new class of matching problems, modeling tuition exchange programs used by colleges in the US as a benefit to faculty members. The most important benefit of participating in a tuition exchange program is that colleges strengthen their compensation package to their faculty and staff at a nominal cost. Participating colleges find that tuition exchange can serve as a strong incentive for top job candidates to accept their offers. Hence, tuition exchange programs help level the playing field for small colleges in hiring and retaining promising faculty. The main concern for each college participating in an exchange is maintaining a balance between the number of exported and imported students. In this paper, we show that decentralized practices used in the field discourage colleges from participation and limit gains from exchange. By intervening the current decentralized practice with a minimal centralization, we can fix most of the woes of this market. This mechanism is not only strategy-proof for students, balanced–efficient, individually rational, and respecting internal priority bylaws of colleges governing the eligibility of sponsored students, it is also the only mechanism that satisfies these properties. Moreover, it cannot be manipulated by colleges through quota manipulation, unlike the current procedures. We also extend TTC to dynamic settings by allowing colleges to run imbalances in a predetermined interval. The interval can be calibrated to have balanced matchings on average.

There also exist tuition co-ops where balancedness is not the main concern. However, extreme imbalance is not desirable and is a danger to the longevity of the program. For these programs, we propose the repeated deferred acceptance mechanism, which is student–strategy-proof and finds the student–optimal weakly stable matching. It also minimizes the aggregate imbalance while respecting stability.

Chapter 2

A Characterization of the Top Trading Cycles Mechanism for the School Choice Problem

2.1 Introduction

In their seminal paper, Abdulkadiroğlu and Sönmez (2003) introduce the school choice problem. Before that paper, in some of the major cities students were assigned to public schools via deficient mechanisms which give high incentives to the students to misreport their true preferences in order to get better allocations. To eliminate the gaming, they propose two competing strategy-proof mechanisms: the Top Trading Cycle (TTC) mechanism and the Deferred Acceptance (DA) mechanism. The TTC mechanism is not only strategy-proof but also Pareto efficient. However, it fails to be fair[1]. On the other hand, the DA mechanism satisfies fairness but fails to be Pareto efficient. When the policy makers decided to adopt one of the two strategy-proof mechanisms, the DA mechanism was selected due to its better features in terms of respecting school district priorities.[2] However, in 2012 New Orleans

[1]Fairness is the natural counterpart of the stability in the school choice context (Balinski and Sönmez, 1999). An allocation is fair if there does not exist a student who prefers another school to his assignment and that school admitted a student with lower priority.

[2]School districts in Boston, New York City and Denver have adopted versions of the DA mechanism.

Recovery School District became the first school district to adopt TTC.

Adoption of the TTC by New Orleans school district shows us that some school districts may value efficiency over fairness. If Pareto efficiency and strategy-proofness are the main objectives of the school districts then TTC can be considered one of the candidates. However, it is not the unique Pareto efficient and strategy-proof mechanism. For instance, the serial dictatorship mechanism also satisfies these two axioms.[3] In this paper, we try to help the policy makers who are willing to adopt a Pareto efficient and strategy-proof mechanism by providing the full characterization of the TTC mechanism. Our characterization is based on Pareto efficiency, strategy-proofness, mutual best along with two axioms that we introduce: resource monotonicity for top-ranked students and weak consistency. We show that TTC mechanism is the unique mechanism satisfying Pareto efficiency, strategy-proofness, mutual best, weak consistency and resource monotonicity for top-ranked students.

"Mutual best"[4] requires that a student be assigned to the school at the top of his preference whenever he has the highest priority at that school. A mechanism is "resource monotonic for top-ranked students" if the assignment of the top-ranked student for a school is not worsened when the number of available seats in that school increases. A mechanism is said to be "weakly consistent" if the removal of a set of agents with their assignments does not

[3]Pycia and Ünver (2011b) provide a class of mechanisms satisfying strategy-proofness and Pareto efficiency in the school choice problem.

[4]Morrill (2012) uses the same axiom in the characterization of TTC in a school choice problem where each school has only one available seat.

affect the assignments of the remaining agents as long as each agent is the top-ranked student for one of the assignment of the removed agent.

Mutual best, weak consistency and resource monotonicity for top-ranked students are weaker forms of fairness, consistency[5] and resource monotonicity[6], respectively. TTC mechanism does not satisfy fairness, consistency and resource monotonicity. In particular, there does not exist a mechanism that is fair, strategy-proof and consistent.[7] Moreover Pareto efficiency and fairness are incompatible.[8] Therefore, we cannot have a mechanism satisfying all of the axioms.[9] Kesten (2006) shows that TTC satisfies fairness, consistency and resource monotonicity if the priority order satisfies strong acyclicity condition. In this paper, we show that TTC is not totally unsuccessful in these three dimensions and none of the Pareto efficient and strategy-proof mechanisms can perform better than TTC in all the three dimensions.

A mechanism which fails to satisfy mutual best, resource monotonicity for top-ranked students and consistency may not meet the demands of both students (families) and school districts. We consider mutual best as a must fairness requirement in the school choice context. For instance, most school

[5]A mechanism is consistent if whenever a set of agents are removed with their assignments then all the remaining agents will be assigned to their initial assignment when we run the mechanism only considering the remaining agents and remaining copies of the objects.

[6]Resource monotonicity requires that if the number of available objects increases then all agents should be affected in the same direction (?).

[7]Alcalde and Barbera (1994) show that DA mechanism is the unique strategy-proof and fair mechanism but it fails to be consistent.

[8]Balinski and Sönmez (1999) show that there does not exist fair and Pareto efficient mechanism.

[9]Serial dictatorship mechanism satisfies four of them. It fails to be fair.

districts give highest priority at a school to a student whose elder sibling is already attending that school and most of the families have preference over keeping their children in the same school (Pathak, 2011). Therefore, both parents and school districts benefit from the mutually best mechanisms . Similarly, resource monotonicity for top-ranked students is a must resource monotonicity requirement. We modify this requirement in two ways. When public goods are allocated, we should not have a decrease in the welfare of any of the agents. Otherwise, providing less and less public goods will be a clear solution for the policy makers. Therefore, we restrict our attention to the mechanisms under which the welfare of agents weakly increases when the number of available objects increases.[10] We also modify the resource monotonicity axiom by only requiring not to have a reduction in the welfare of the top-ranked student for the school whose number of seats has increased. Therefore any resource monotonic mechanism under which welfare of the agents weakly increase with an increase in the number of available objects satisfies resource monotonicity for top-ranked students. Consistency is a desired property in the school choice context where the assignment process for different types of schools are done separately. For instance, in New York City the assignment of exam and mainstream schools are done separately (Abdulkadiroğlu, Pathak, and Roth, 2009). Therefore, running a consistent mechanism will prevent the request of remaining agents for another run when the other agents are removed with their assignments.

[10]Kojima and Ünver (2010) define resource monotonicity similarly.

Although, mutual best and resource monotonicity for top-ranked students axioms are enough to prove our uniqueness result, the TTC mechanism satisfies stronger forms of these two axioms. TTC respects the priority of student i for school s if the number of students with higher priority for school s is less than the number of available seats in that school. Moreover, if the policy makers and families are only sensitive to priority violation in the upper priority groups then TTC can be considered to have a good performance in terms of respecting priorities. Under TTC mechanism, the students who are ranked at the top q of the priority order of school s cannot be made worse off due to the increase in the number of available seats from q to q'.

This is the first paper characterizing TTC mechanism in the school choice context where each school may have more than one available seat. Abdulkadiroğlu and Che (2010) and Morrill (2012) provide alternative characterizations of TTC mechanism in the school choice context where each school is restricted to have only one available seat. Abdulkadiroglu and Che show that TTC mechanism is the only mechanism that is Pareto efficient, strategy-proof and recursively respects top priorities.[11] Morill characterizes the TTC mechanism in two different ways. He first shows that TTC is the unique mechanism which is strategy-proof, Pareto efficient, and independent of irrelevant rankings[12] and satisfies mutual best. He also demonstrates that TTC is

[11] A mechanism respects top priorities if an agent is assigned an object, then the agent that is top-ranked by that object should not be assigned to a worse object than that object.

[12] A mechanism is independent of irrelevant rankings if whenever the ranking of an agent at an object's priority order does not affect the assignment of that agent then it does not

the unique mechanism satisfying Pareto efficiency, independence of irrelevant rankings, weak Maskin monotonicity and mutual best. Results of these two papers do not hold in the school choice problem where schools may have more than one available seat (Morrill, 2012). Sönmez and Ünver (2010) provide the characterization of the you request my house-I get your turn (YRMH-IGYT) mechanisms in the house allocation problems with existing tenants (Abdulkadiroğlu and Sönmez, 1999). They show that YRMH-IGYT mechanism is the unique mechanism satisfying Pareto efficiency, strategy-proofness, individual rationality, weak neutrality[13] and consistency.[14] Pycia and Ünver (2011a) introduce a class of mechanism called trading cycles mechanisms and show that in the house allocation problem a mechanism is individually rational, Pareto efficient, group strategy-proof if an only if it is a trading cycles mechanism.[15] Pycia and Ünver (2011b) also analyze trading cycles mechanism in the school choice environment where each school may have more than one available seat and show that trading cycles mechanisms are Pareto efficient and strategy-proof.

The rest of the paper is organized as follows: In Section 2 we introduce the model and properties of mechanisms. In Section 3 we describe the TTC mechanism. We present our main results in Section 4. In Section 5 we show

affect the assignment of all the other agents.

[13]If a mechanism satisfies weak neutrality then the outcome of that mechanism will not depend on the names of the unoccupied objects.

[14]Sönmez and Ünver (2010) also consider a weaker version of consistency in the house allocation problem with existing tenants.

[15]The TTC mechanism belongs to the class of the trading cycles mechanisms.

the independence of axioms used in our main results. A brief conclusion is given in the final section.

2.2 Model

A school choice problem is a list $[I, S, q, P, \succ]$ where

- I is the set of students,
- S is the set of schools,
- $q = (q_s)_{s \in S}$ is the quota vector where q_s is the number of available seats in school s,
- $P = (P_i)_{i \in I}$ is the preference profile where P_i is the strict preference of student i over the schools including no-school option,
- $\succ = (\succ_s)_{s \in S}$ is the priority profile where $\succ_s$ is the priority relation of school s over I.

We denote the no-school option with $s_\emptyset$ and $q_{s_\emptyset} = \infty$. Let R_i be the at-least-as-good-as relation associated with the strict preference order P_i and for all $s,\ s' \in S \cup s_\emptyset$ sR_is' if and only if $s = s'$ or sP_is'. We assume that there are no ties in the priority profiles of schools.[16]

A matching is a function $\mu : I \to S \cup s_\emptyset$ such that $\mu(i) = s$ and $\mu(i) = s'$ if only if $s = s'$. If $\mu(i) = s_\emptyset$ then student i is unassigned. In a matching μ,

[16]School districts mostly use random tie breaking rules.

the number of students assigned to a school s cannot exceed the total number of available seats in school s. Let $\mathcal{M}$ be the set of all possible matchings.

A mechanism is a procedure which selects a matching for each problem. That is, a mechanism φ takes the preference profile of the students, the priority order of students for schools, the quota vector, then selects a matching for every problem. The matching selected by mechanism φ in problem $[I, S, q, P, \succ]$ is denoted by $\varphi[I, S, q, P, \succ]$. Let $\varphi[I, S, q, P, \succ](i)$ denote the assignment of student $i \in I$ by mechanism φ for problem $[I, S, q, P, \succ]$.

Student i *strictly prefers* matching μ to matching μ' if he strictly prefers $\mu(i)$ to $\mu'(i)$, $\mu(i) P_i \mu'(i)$. A matching μ is *Pareto efficient* if there does not exist a matching $\mu' \in \mathcal{M}$ in which each student is not worse off and at least one student is strictly better off. More formally, matching μ is Pareto efficient if there does not exist a matching $\mu' \in \mathcal{M}$ where $\mu'(i) R_i \mu(i)$ for each $i \in I$ and $\mu'(j) P_j \mu(j)$ at least for one $j \in I$. A mechanism φ is *Pareto efficient* if for all problems it selects a Pareto efficient matching.

A mechanism φ is *strategy-proof* if it is (weakly) dominant strategy for all students to tell their preferences truthfully. Formally, a mechanism φ is strategy-proof if for every preference profile P and P_i' $\varphi[I, S, q, P, \succ](i) R_i \varphi[I, S, q, (P_i', P_{-i}), \succ](i)$ for all student $i \in I$. Here, P_{-i} represents the true preference profile of students except i.

Let $t_i^{\succ}$ be the set of schools ranking i over all other students under priority profile $\succ$. Formally, $t_i^{\succ} = \{s \in S | i \succ_s j \ \forall j \in I \setminus i\}$. A mechanism ϕ

is *mutually best* if whenever there exists $s \in t_i^{\succ}$ such that $q_s > 0$, sP_is' for all $s' \in S \setminus \{s\}$ with $q_{s'} > 0$ then $\phi[I,\ S,\ q,\ P,\ \succ](i) = s$ for all $i \in I$.[17]

A mechanism ϕ is *resource monotonic* if for all $s \in S$, all $q'_s \leq q_s$ either for all $i \in I$, $\phi[I, S, q, P, \succ](i) R_i\ \phi[I, S, (q'_s, q_{-s}), P, \succ](i)$ or for all $i \in I$ $\phi[I, S, (q'_s, q_{-s}), P, \succ](i) R_i \phi[I, S, q, P, \succ](i)$.[18] I use a different version of resource monotonicity. Intuitively, if student i has the highest priority for school s then his welfare should not be worsened when the number of seats in school s increases. I formally define resource monotonicity for top-ranked students as follows: A mechanism ϕ is *resource monotonic for top-ranked students* if for all $i \in I$ and all $q'_s \geq q_s > 0$ where $s \in t_i^{\succ}$ $\phi[I, S, (q'_s, q_{-s}), P, \succ](i) R_i \phi[I, S, q, P, \succ](i)$.

Before introducing our consistency axiom we need additional notation.

For any school $s \in S$, priority order $\succ_s$, and a set of students $J \subset I$, let $\succ_s^J$ be the restriction of priority order $\succ_s$ to students in J. Formally, for any $i, j \in J$ $i \succ_s^J j$ if and only if $i \succ_s j$. Let $\succ^J = (\succ_s^J)_{s \in S}$ and $\succ^{-J} = (\succ_s^{I \setminus J})_{s \in S}$.

Given a problem $[I, S, q, P, \succ]$, a set of students $J \subset I$, and a quota profile $\tilde{q} \leq q$ we say $\left[J, S, \tilde{q}, P_{-J}, \succ^J\right]$ is the restriction of the problem $[I, S, q, P, \succ]$ to students in J and quota profile $\tilde{q}$.[19]

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[17] Morrill (2012) defines mutual best similarly. Different from the definition of Morrill (2012) our definition takes into consideration the fact that some schools may not have available seats.

[18] See **?**, Ehlers and Klaus (2003) and Kesten (2009) for related results.

[19] Similar notation is used in Sönmez and Ünver (2010).

A mechanism is consistent if whenever a set of students are removed with their assignments then all the remaining students will be assigned to their initial assignment when we run the mechanism only considering the remaining students and objects.[20] Formally, a mechanism ϕ is *consistent* if for any problem $[I, S, q, P, \succ]$, when we remove a set of students $J \subset I$ together with their assignments $\phi[I, S, q, P, \succ](J)$, then for any $i \in I \setminus J$

$$\phi[I \setminus J, S, \tilde{q}, P_{-J}, \succ^{-J}](i) = \phi[I, J, q, P, \succ](i)$$

where $\tilde{q}_s$ is the number of available seats remaining in school s.

In this paper, we introduce a weaker version of the consistency axiom.[21] A mechanism satisfies weak consistency if whenever we remove a set of students with their assignment such that the student with the highest priority for one of the removed student's assignment is also another removed student then the assignments of the remaining students do not change.

A mechanism ϕ is *weakly consistent* if for any problem $[I, S, q, P, \succ]$, when we remove a set of students $J \subset I$ together with their assignments $\phi[I, S, q, P, \succ](J)$ satisfying $|t_j^{\succ} \cap \phi[I, S, q, P, \succ](J)| = 1$ for each $j \in J$, then for any $i \in I \setminus J$

[20] See **?** and Ergin (2000) for related results.

[21] Sönmez and Ünver (2010) also modifies the definition of the consistency axiom. In that paper, they characterize YRMH-IGYT in the house allocation problem with existing tenants. YRMH-IGYT also fails to satisfy the consistency axiom but satisfies the modified version defined in that paper.

$$\phi[I \setminus J, S, \tilde{q}, P_{-J}, \succ^{-J}](i) = \phi[I, J, q, P, \succ](i).$$

Our restriction on the set of students and seats removed is simple. It is easy to see that any mechanism which is consistent based on the traditional definition satisfies the weaker form of it that we define here.

2.3 Top Trading Cycles Mechanism

In the school choice context, the TTC mechanism was first introduced by Abdulkadiroğlu and Sönmez (2003). It was based on the Gale's top trading cycles algorithm (Shapley and Scarf, 1974). It is a direct mechanism and for any given problem $[I, S, q, P, \succ]$ it works iteratively in a number of steps:

Top Trading Cycles Mechanism (TTC):

Step 0: Assign a counter to each school and set it to the quota of each school. If the counter of a school is zero, then that school is removed.

Step 1: Each student points to his most preferred school among the remaing ones. Each remaining school points to the top-ranked student in its priority order. School $s_\emptyset$ points to all students pointing to it. Due to the finiteness there is at least one cycle.[22] Assign every student in the cycles to the school he points to and remove him. The counter of each school in a cycle is reduced by one and if it reduces to zero, the school is also removed.

[22] A cycle is an ordered list of distinct schools and distinct students $(s_1, i_1, s_2, ..., s_k, i_k)$ where s_1 points to i_1 , i_1 points to s_2 , ... , s_k points to i_k , i_k points to s_1 .

In general,

Step k: Each student points to his most preferred school among the remaining ones. Each remaining school points to the student with the highest priority among the remaining ones. School $s_\emptyset$ points to all students pointing to it. There is at least one cycle. Assign every student in the cycles to the school he points to and remove him. The counter of each school in a cycle is reduced by one and if it reduces to zero, the school is also removed. Our definition of TTC mechanism differs from the one provided in Abdulkadiroğlu and Sönmez (2003) by considering the posibility that some schools may not have available seats.

The algorithm terminates when all students are assigned.

We illustrate the dynamics of TTC mechanism in the following example.

Example 6 *Let* $S = \{s_1, s_2, s_3, s_4\}$, $I = \{i_1, i_2, i_3, i_4, i_5\}$ *and* $q = (1, 1, 1, 2)$. *The preferences of students and priorities are as follows:*

$$i_1 : s_1 P_{i_1} s_2 P_{i_1} s_3 P_{i_1} s_4 \qquad s_1 : i_5 \succ_{s_1} i_3 \succ_{s_1} i_4 \succ_{s_1} i_2 \succ_{s_1} i_1$$

$$i_2 : s_2 P_{i_2} s_1 P_{i_2} s_4 P_{i_2} s_3 \qquad s_2 : i_3 \succ_{s_2} i_1 \succ_{s_2} i_4 \succ_{s_2} i_2 \succ_{s_2} i_1$$

$$i_3 : s_1 P_{i_3} s_3 P_{i_3} s_4 P_{i_3} s_2 \qquad s_3 : i_3 \succ_{s_3} i_2 \succ_{s_3} i_4 \succ_{s_3} i_1 \succ_{s_3} i_5$$

$$i_4 : s_3 P_{i_4} s_4 P_{i_4} s_1 P_{i_4} s_2 \qquad s_4 : i_1 \succ_{s_4} i_3 \succ_{s_4} i_2 \succ_{s_4} i_5 \succ_{s_4} i_4$$

$$i_5 : s_4 P_{i_5} s_1 P_{i_5} s_2 P_{i_5} s_3$$

Step 0: All schools have available seats and we set counters to the quotas of the schools.

Step 1: Each students points to his most preferred school and each schools points to the student with the highest priority. There is only one cycle: (s_1, i_5, s_4, i_1). We assign each student in the cycle to the school he points to and remove him: $\mu(i_1) = s_1$ and $\mu(i_5) = s_4$. We also reduce the counter of each school in the cycle and remove only s_1 since its counter reduces to zero.

Step 2: Each remaining students points to his most preferred remaining school and each remaining schools points to the student with the highest priority among the remaining ones. There is only one cycle: (s_3, i_3). We assign the student in the cycle to the school he points to and remove him: $\mu(i_3) = s_1$. We also reduce the counter of the school in the cycle and remove it, s_3, since its counter reduces to zero.

Step 3: Each remaining students points to his most preferred remaining school and each remaining schools points to the student with the highest priority among the remaining ones. There is only one cycle: (s_2, i_4, s_4, i_2). We assign each students in the cycle to the school he points to and remove him: $\mu(i_2) = s_2$ and $\mu(i_4) = s_4$. We also reduce the counter of each school in the cycle and remove only both of them since their counter reduce to zero.

The mechanism terminates since all students are assigned.

2.4 Results

In the following theorem, we show that TTC is Pareto efficient, strategy-proof, weakly consistent, resource monotonic for top-ranked students and mutually best. Moreover, there does not exist another mechanism satisfying all these axioms. We prove it in the Appendix B.

Theorem 11 *In school choice problem TTC is the unique mechanism satisfying*

- *Pareto efficiency*
- *Strategy-proofness*
- *Weak consistency*
- *Resource monotonicity for top-ranked students*
- *Mutual best.*

In the next section, we show that there always exist another mechanism satisfying only four of the five axioms.

Abdulkadiroğlu and Che (2010) also use Pareto efficiency and strategy-proofness in their characterization. In addition to Pareto efficieny and strategy-proofness they use recursively respecting top priorities. One can suspect whether weak consistency, resource monotonicity for top-ranked students and mutual best implies recursively respecting top priorities. In the following example we show that this is not true.

Example 7 *Boston mechanism satisfies consistency and resource monotonicity (Kojima and Ünver, 2010). Theorefore, it is weakly consistent and resource* *monotonic for the top ranked agents. Moreover, in the first step of the Boston mechanism if a student applies to his most popular school for which he has the top priority, then he will be assigned to that school. Therefore, it satisfies mutual best.*

Now consider the following example. There are three schools, $S = \{s_1, s_2, s_3\}$*, and three students,* $I = \{i_1, i_2, i_3\}$*. Each school has one available seat. The prefernce profile and priority structure are given as:*

$$i_1 : s_1 P_{i_1} s_2 P_{i_1} s_3 \qquad s_1 : i_3 \succ_{s_1} i_2 \succ_{s_1} i_1$$

$$i_2 : s_2 P_{i_2} s_1 P_{i_2} s_3 \qquad s_2 : i_1 \succ_{s_2} i_2 \succ_{s_2} i_3$$

$$i_3 : s_1 P_{i_3} s_3 P_{i_3} s_2 \qquad s_3 : i_3 \succ_{s_3} i_2 \succ_{s_3} i_1$$

The outcome of Boston mechanism is: $\mu(i_1) = s_3$*,* $\mu(i_2) = s_2$ *and* $\mu(i_3) = s_3$*. Matching* μ *does not respect top priorities since* i_2 *is assigned to* s_2 *and the student* i_1 *who has the top priority for* s_2 *is assigned to a worse school than* s_2*. Therefore, Boston mechanism does not (recursively) respect top priorities.*

Mutual best can be considered as a very weak fairness requirement and satisfying it may not make a mechanism more desirable. In the following proposition, we show that TTC mechanism satisfies much stronger fairness requirement.

Proposition 15 *Under TTC mechanism, each student weakly prefers his assignment to each school* s *for which he is ranked at the top* q_s *portion of that* *school's priority order.*

Suppose not. Let student i's rank for school s be $r < q_s$ and he be assigned to school s' such that sP_is'. School s will start pointing student i after $r-1$ students are assigned to it if i is not assigned in an earlier step. First consider the case that i is not assigned before s points him. School s will keep pointing i until he is removed. Therefore, i will be assigned to s whenever he points to that school. Now consider the case that i is assigned before s points to him. In this case, i should be assigned to a better school and he never points to s.

We can also show that TTC mechanism satisfies a general form of resource monotonicity for top-ranked student.

Proposition 16 *When the number of available seats in school s is increased from q_s to $\tilde{q}_s$, keeping everything else the same, then TTC mechanism assigns top q_s students in school s's priority order to weakly better schools.*

We refer to the proof of Theorem 11. The part that we prove TTC mechanism is resource monotonic for top-ranked students can be extended for top q_s students. It follows from the fact that the first $q \leq q_s$ seats of school s cannot be filled before top q students in school s's priority order are removed.

So far, we show that TTC mechanism outperforms other strategy-proof and Pareto efficient mechanisms. Some school districts consider fairness as the most important concern and these districts select DA mechanism instead of the TTC mechanism. In the rest of this section, we focus on the fairness and the performance of the TTC in terms of respecting priorities.

In the most of the school districts, priority structure is determined based on some exogenous rules. For instance, Boston school district gives the highest priority for a school to the students living in the same walk zone and having a sibling attending that school.[23] The second priority is given to students having a sibling attending that school but living outside the walk zone of that school. Students who are only living in the same walk zone have the third priority and the fourth priority is given to the remaining students. Ties between students in the same priority group is broken by random lottery. That is, the priority structure, $\succ$, in any problem is determined based on the priority groups and random draw. Public policy makers and families might give more importance respecting priorities in the upper priority groups (**?**). In Proposition 17, we show that TTC is successful at respecting priorities in the upper priority groups under some realistic conditions. Before presenting our results we need some notation.

Suppose there are n priority groups and respecting priorities in the first n^* priority group is more important. Let $G_i : S \to \mathcal{N}$ be a function and $G_i(s)$ be the priority group that student i belongs to for school s. We say student i's preference P_i is **perfectly correlated with** the priority groups if the following condition holds: if $G_i(s) < n^*$ and $G_i(s) < G_i(s')$ then sP_is'. A

[23] This priority group is known as sibling-walk zone priority.

preference profile $P = (P_i)_{i \in I}$ is perfectly correlated with the priority groups if each student's preference is perfectly correlated with the priority groups. As an example, suppose the first priority group (sibling-walk zone) in Boston is given more importance than the others. Then the preference profile of the students is perfectly correlated with the priority groups if each student having sibling-walk zone priority in some school ranks one of the schools for which he has sibling-walk zone priority at the top of his preference list.

Now we are ready to present our result on the performance of the TTC mechanism in terms of respecting priorities.

Proposition 17 *Let π be the outcome of TTC mechanism in problem $[I, S, q, P, \succ]$. There does not exist a student and school pair (i, s) such that $G_i(s) < n^*$, $sP_i\pi(i)$, there exists another student j assigned to s and $i \succ_s j$ if any one of the following conditions holds:*

(a) The total number of students in the first n^* priority class of each school s is less than or equal to q_s.

(b) Preference profile P is perfectly correlated with the priority groups.

Table 2.1 shows the precentage of schools (grade K2) in Boston Public School System where the number of applicants with sibling priority is more than the number of available seats for the entering class. This data shows that TTC respects the sibling-walk zone and sibling priorities of almost all students.

Year	2006	2007	2008	2009	2010
Percentage	0.8696%	2.521%	2.5641%	0.8929%	4.878%

Table 2.1: Percentage of School with Excess Sibling Priority Applicants

2.5 Independence of Axioms

Below we show the independence of axioms mentioned in Theorem 11.

- *Strategy-proof, weakly consistent, resource monotonic for top-ranked students, and mutually best, but not Pareto efficient:* Consider the following problem. Two schools $S = \{a, b\}$ with one available seat and two students $I = \{\mathbf{1}, \mathbf{2}\}$. Let the preference profile P and priority order $\succ$ be

$P_{\mathbf{1}}$	$P_{\mathbf{2}}$
b	a
a	b
$s_\emptyset$	$s_\emptyset$

$\succ_a$	$\succ_b$
$\mathbf{1}$	$\mathbf{2}$
$\mathbf{2}$	$\mathbf{1}$

  Let mechanism ψ assign $\mathbf{2}$ to b and $\mathbf{1}$ to a. Let ψ select the same assignment in the above problem independent of preferences. For all other problems, ψ selects the same matching as TTC mechanism. Mechanism ψ fails to be Pareto efficient and satisfies other 4 properties.

- *Strategy-proof, weakly consistent, resource monotonic for top-ranked students, and Pareto efficient, but not mutually best:* Serial dictatorship mechanism is strategy-proof, (weakly) consistent, and Pareto-efficient. Moreover, when the number of available seats in a school is increase all students' welfare weakly improve. That is, it satisfies more generalized version of the resource monotonicity for top-ranked students. However, it fails to be mutually best.

- *Strategy-proof, weakly consistent, Pareto efficient, and mutual best mechanism, but not resource monotonic for top-ranked students:* Consider the following problem: Two schools $S = \{a, b\}$ with one available seat and three students $I = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$. Let the preference profile P and priority order $\succ$ be

$P_{\mathbf{1}}$	$P_{\mathbf{2}}$	$P_{\mathbf{3}}$	$\succ_a$	$\succ_b$
b	b	a	**1**	**3**
a	a	b	**2**	**1**
$s_\emptyset$	$s_\emptyset$	$s_\emptyset$	**3**	**2**

Let mechanism ψ assign **3** to a and **1** to b in this problem. If the number of available seats in school a is increased to 2 then ψ assigns **1** and **3** to a and **2** to b. Let ψ select the same assignment in the above problem where a has two available seats and **1** ranks a above $s_\emptyset$ and assign **1** to $s_\emptyset$ if he ranks a below $s_\emptyset$. For all other problems ψ selects the same matching as TTC mechanism. Mechanism ψ fails to be resource monotonic for top-ranked students and satisfies other 4 properties.

- *Strategy-proof, Pareto efficient, mutually best mechanism, resource monotonic for top-ranked students but not weakly consistent:* Consider the following problem. Three schools $S = \{a, b, c\}$ with one available seat and three students $I = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$. Let the preference profile P and priority order $\succ$ be

$P_{\mathbf{1}}$	$P_{\mathbf{2}}$	$P_{\mathbf{3}}$
c	a	a
a	b	b
b	c	c

$\succ_a$	$\succ_b$	$\succ_c$
1	**1**	**1**
2	**2**	**2**
3	**3**	**3**

Let mechanism ψ assign **1** to c and **2** to b and **3** to a in this problem. Let ψ select the same matching as long as **1** and **3** submit the same preferences and **2** ranks b over $s_{\emptyset}$. If we remove **1** with his assignment then **2** is assigned to a and **3** is assigned to b. For all other problems ψ selects the same matching as TTC mechanism. Mechanism ψ fails to be consistent and satisfies other 4 properties.

- *Pareto efficient, mutually best mechanism, resource monotonic for top priority students and consistent but not strategy-proof:* The Boston mechanism is Pareto efficient, resource monotonic and consistent (Kojima and Ünver, 2010). Moreover, in the first step of the Boston mechanism when a student applies to his most popular school for which he has the highest priority he will be assigned to that school. Therefore it satisfies mutual best. The Boston mechanism fails to be strategy-proof (Abdulkadiroglu and Sonmez, 2003) and satisfies other 4 properties.

2.6 Conclusion

TTC mechanism has been studied extensively in the market design literature. It and its variants have been proposed as one of the best alternatives in many matching markets including public school choice systems, on-campus housing and the kidney exchange programs. However, TTC mechanism has never been characterized for the cases where objects have a capacity greater than one, i.e. school choice problem. In this paper, we provide the first characterization of the TTC mechanism in the school choice problem. Our characterization will help the school districts choose between strategy-proof and Pareto efficient mechanisms. In particular, TTC mechanism is the unique strategy-proof and Pareto efficient mechanism satisfying mutual best, weak consistency and resource monotonicity for top-ranked students.

We also focus on the performance of the TTC mechanism in terms of respecting priorities. We show that TTC mechanism respects priorities in the upper priority classes. If the policy makers and families are only sensitive for the priority violations in the upper priority classes then TTC mechanism will meet their needs.

Chapter 3

A 1.5-Sided Matching Problem: The Cost-Efficient House Allocation

3.1 Introduction

The house allocation problem with existing tenants was introduced by Abdulkadiroğlu and Sönmez (1999).[1] In their seminal paper, they compare the mechanisms used for allocating campus housing to college and graduate students in the US. They show that the mechanisms in use cause avoidable welfare losses for students, and they propose the TTC mechanism instead. The TTC mechanism is not only Pareto efficient but also strategy-proof and individually rational. However, the TTC mechanism fails to be fair (stable).[2]. TTC is not the only alternative discussed in the literature. Guillen and Kesten (2012) show that the New House 4 (NH4) mechanism, which is used at MIT, is equivalent to the DA mechanism, where each existing tenant is given the highest priority in his current house.[3] Hence, this mechanism is individually

[1] The problem consists of existing tenants, newcomers, and occupied and vacant houses. Each agent has preferences over the houses. Each existing tenant owns the house he is living in. Each house ranks agents based on a single ordering.

[2] Fairness simply requires respecting priorities (Balinski and Sönmez, 1999). In the HAP-wET literature fairness is not considered as an important issue because the priority structure is determined by a draw. In this paper, we construct the priority order of a house by giving the highest priority to the occupant and rank all the other agents according to the draw.

[3] In the rest of the paper we consider this variant of DA mechanism.

rational, strategy proof, and fair, but fails to be Pareto efficient. On the other hand, its outcome Pareto dominates any other fair matching. In the literature, these two mechanisms, DA and TTC, have been compared based on their performance along various dimensions (see Chen and Sönmez, 2002; Guillen and Kesten, 2012; Kesten, 2006). TTC can be considered the best mechanism if Pareto efficiency is valued over fairness. On the other hand, DA can be considered the best mechanism if fairness is valued over efficiency. Although housing offices can replace their current deficient mechanism with one of these two alternatives, most colleges still use a variant of the serial dictatorship mechanism. For instance, Random Serial Dictatorship with Squatting Rights (RSDwSR) is a widely used variant of the serial dictatorship mechanism.[4] Under RSDwSR, existing tenants have the right to keep their current houses, but if they want to apply for a new house they have to give up their property rights for their current houses. The possibility of being assigned to a worse house prevents many existing tenants from participating and applying for a new house. As Abdulkadiroğlu and Sönmez (1999) show, this causes welfare losses for students. Some universities make life more difficult for the existing tenants. For instance, at the University of Texas if an existing tenant wants to move to another house he has to give up his property rights on his house in order to apply for a new house. Moreover, he is placed at the bottom of the waiting list for a new house. When the supply of houses is limited,

[4]Carnegie Mellon, Duke, Northwestern, the University of Michigan, and the University of Pennsylvania employ the RSDwSR mechanism.

the existing tenant who applies for a new house may not be assigned to any house. As a consequence, existing tenants keep their current houses and do not participate in the allocation process. On the other hand, any individually rational mechanism guarantees that any existing tenant will be assigned a new house at least as good as his current house. Therefore, the participation rate of the existing tenants should be higher under the individually rational mechanisms. Then why don't housing offices act benevolently and adopt a mechanism that leads to welfare gains by increasing the participation rate?

A possible reason for sticking to RSDwSR and not replacing it with an individually rational mechanism is that RSDwSR decreases the cost to housing offices by not giving an incentive to the existing tenants to participate. Here, cost can be the administrative work of a move, the cleaning/maintenance cost of an occupied house whose tenant moved to another house, or the opportunity cost of not renting a house for a given time interval. Therefore, keeping the existing tenants in their current houses is preferable for the housing office.

In this paper we compare the DA and TTC mechanisms based on their cost to the housing office. To our knowledge this is the first comparison of the two mechanisms that takes the preferences of the housing office (central authority) into account. We assume that a mechanism in which more existing tenants are assigned to their current house is preferable for the housing offices due to the lower cost of moving. A housing office that aims to adopt one of the individually rational and strategy-proof mechanisms may find our comparison helpful. We first show that RSDwSR can be more costly than both the TTC and DA mechanisms (Example 8). Therefore, replacing RSDwSR with one of the two mechanisms can be beneficial for the housing offices in addition for the students. Then, we show that the DA mechanism is less costly if there exists only one existing tenant (Proposition 18). When there is more than one existing tenant there is no dominance between the two mechanisms in terms of cost efficiency[5] (Example 10). However, for any priority order and set of students there always exists a preference profile such that the outcome of TTC is more costly than the outcome of DA and the reverse is not true (Proposition 19). Moreover, whenever TTC is less costly than the DA mechanism, then it fails to be fair. On the other hand, it is possible that DA selects a Pareto-efficient outcome that is less costly than the outcome of TTC (Proposition 20).

Since there is no clear-cut dominance between TTC and DA in terms of cost efficiency, we use simulations to compare the performance of the two mechanisms. In the simulations we calculate the average number of existing tenants who keep their own houses and we count the number of runs in which one of the mechanisms assigns strictly more existing tenants to their own houses. Our simulation results show that in almost all cases DA performs better than TTC from the point of view of the housing office.

We include the housing office in the welfare analysis and redefine the Pareto efficiency axiom. Based on our new Pareto efficiency definition, every

[5]By cost efficient we mean less costly.

fair matching is also Pareto efficient (Theorem 12). There always exist both a best and worst fair matching for agents.[6] We show that the best fair matching for students is the least preferred fair matching for the housing office (Proposition 22). In this setting we can also weakly rank the Pareto efficient matchings based on the housing office's welfare. The Pareto efficient matchings in which all the existing tenants are assigned to their own houses comprise the set of the best Pareto efficient matchings for the housing office. Whenever the TTC mechanism selects a Pareto efficient matching that is one of the best Pareto efficient matchings for the housing office, then the DA mechanism will select the same matching. However, TTC may fail to select a Pareto efficient matching that is one of the best Pareto efficient matching for the housing office when the DA mechanism selects it (Proposition 23). When different orderings are used for the houses, there does not exist a fair and Pareto efficient mechanism (Theorem 13). In contrast to the one-sided problems, the DA mechanism does not always select the Pareto efficient and fair matching whenever it exists. Finally, we provide conditions on the preferences of the housing office guaranteeing that every fair matching also satisfies Pareto efficiency (Theorem 14).

It is natural to think that the housing office can decrease the number of moves by changing its priority structure. The obvious way is to give lower priority to the existing tenants than to the newcomers in all houses except their own houses. However, this is not true (see Example 2). That is, the

[6] A fair matching is the best fair matching for agents if every agent likes it at least as well as any other fair matching. A fair matching is the worst fair matching for agents if every agent likes any other fair matching at least as well as that matching.

housing office cannot minimize its costs by just giving lower priority to the existing tenants under DA or TTC.

By including the housing office (central authority) in the welfare analysis we help to close the gap between two-sided matching problems (Gale and Shapley, 1962; Roth, 1985) and one-sided matching problems (Abdulkadiroğlu and Sönmez, 1999, 2003). HAPwET has been studied as a one-sided matching problem (Abdulkadiroğlu and Sönmez, 1999; Sönmez and Ünver, 2010). In two-sided matching problems, agents in both sides of the market are included in the welfare analysis. That is, we can turn HAPwET into a two-sided matching problem by taking the priority orders as the preferences of the houses and including the houses in the welfare analysis. We also increase the number of the sides of the market by including the housing office in the welfare analysis. Under our framework, some of the results that hold in two-sided problems but not in one-sided problems become valid. For instance, in our framework fairness implies Pareto efficiency, as in the two-sided problems. However, including the housing office in the welfare analysis is not equivalent to including all houses and turning the problem into a two-sided matching problem (see Theorem 13).

Considering fairness as a desirable feature in the house allocation problem can be criticized. Earlier works mainly focus on problems where the ordering used in the allocation is drawn randomly and violation of the ordering is not considered a problem. However, in many colleges the ordering used in the allocation is not determined randomly. For instance, students may be ranked based on their seniority or GPA. When the students are ordered based on predetermined rules by the colleges any violation of the ordering can be considered a problem. Therefore, fairness should be taken as a desired property.

The rest of the paper is organized as follows: In Section 2 we introduce the model and properties of the mechanisms. In Section 3 we describe the competing mechanisms. We present our main results in Section 4. In Section 5 we show our simulation results. A brief conclusion is given in the final section.

3.2 Model

The house allocation problem with existing tenants consists of

1. a finite set of existing tenants I_E,
2. a finite set of newcomers I_N,
3. a finite set of occupied houses $H_O = \{h_i\}_{i \in I_E}$,
4. a finite set of vacant houses H_V,
5. a list of preference relations $P = (P_i)_{i \in I_E \cup I_N}$,
6. an ordering, f, over all agents.

Let $I = I_E \cup I_N$ and $H = H_O \cup H_V$ denote the set of agents and houses, respectively. Let h_0 denote the null house: the option of being unassigned.

Let $f(k)$ be the k^{th} ranked agent in f . Each existing tenant $i \in I_E$ has property rights for house h_i. Let $\theta = (\theta_i)_{i \in I}$ be the ownership profile where $\theta_i = h_i$ if $i \in I_E$ and $\theta_i = h_0$ if $i \in I_N$. The exogenous ordering f is randomly chosen from a given distribution of orderings. The distribution of orderings can be uniform or be dictated by seniority or academic performance.[7] Given ordering f and the ownership profile we construct the priority order for each house $h \in H$, $\succ_h$, as (1) for each $h \in H_V \cup h_0$, $i \succ_h j$ if $f^{-1}(i) < f^{-1}(j)$ and (2) for each $h_i \in H_O$, $i \succ_{h_i} j$ for all $j \in I \setminus \{i\}$ and for each $k, l \in I \setminus \{i\}$ $k \succ_{h_i} l$ if $f^{-1}(k) < f^{-1}(l)$. Let $\succ = (\succ_h)_{h \in H}$.[8]

The preference relation P_i for each agent $i \in I$ is assumed to be strict. We denote the weak preference relation associated with P_i by R_i. For expositional simplicity, we assume that the null house, h_0, is the least preferred house for each agent.[9]

We fix the set of agents and houses and the orders and define a house allocation problem with preference profile $[P]$.

A matching $\mu : I \to H \cup h_0$ is a function such that every agent is assigned to one house (including h_0) and only h_0 can be assigned to more than one agent. Let $\mathcal{M}$ be the set of matchings.

[7]For instance, if the distribution of orderings is dictated by seniority, then randomly selecting an exogenous ordering is equivalent to breaking the ties betweens the student in the same cohort based on a random rule.

[8]Given a house allocation problem with existing tenants, Guillen and Kesten (2012) construct the priority order for each house by using the ownership profile and ordering in a similar way.

[9]This assumption has been commonly used in earlier papers studying the house allocation problem with exiting tenants.

A matching $\mu \in \mathcal{M}$ **Pareto dominates** another matching $\nu \in \mathcal{M}$ if $\mu(i)\ R_i\ \nu(i)$ for all $i \in I$ and $\mu(j)\ P_j\ \nu(j)$ for some $j \in I$. A matching is **Pareto efficient** if it is not Pareto dominated by any other matching.

A matching μ is **individually rational** if each existing tenant weakly prefers his assignment in μ to the house that he has been occupying. That is, μ is individually rational if $\mu(i) R_i h_i$ for all $i \in I_E$.[10]

A matching μ is **fair** if, whenever an agent i prefers house h to his assignment $\mu(i)$, then there exists an agent $j \in I$ assigned to j in μ and j has higher priority than i for house h. Formally, a matching μ is fair if whenever $hP_i\mu(i)$ then $\mu^{-1}(h) = j$ and $j \succ_{\mu(j)} i$. It is easy to see that if a matching is fair then it satisfies individual rationality.[11]

A **mechanism** is a systematic way of selecting a matching for each problem. Let φ be a mechanism; then the matching selected by φ in problem $[P]$ is denoted by $\varphi[P]$ and the assignment of agent $i \in I$ is denoted by $\varphi[P](i)$.

A mechanism is Pareto efficient if it selects Pareto efficient matching in every problem. A mechanism φ is individually rational if the outcome of φ for every problem is individually rational. A mechanism φ is fair if it selects a fair matching in every problem.

[10] Note that we do not require $\mu(i) R_i h_0$ for all $i \in I$ since h_0 is assumed to be the least preferred house for each agent.

[11] Note that, if we consider the priority order of each house h as the preference of that house then the stability, which is highly discussed in the two-sided matching literature (Roth and Sotomayor, 1990), and our fairness notions are equivalent.

A mechanism φ is **strategy-proof** if reporting true preferences is a weakly dominant strategy for each agent in the preference revelation game; that is, there does not exist a problem $[P]$, an agent $i \in I$, and a preference relation P'_i such that $\varphi[P'_i, P_{-i}]\ P_i\ \varphi[P]$. Here, $P_{-i} = (P_j)_{j \in I \setminus \{i\}}$.

All these axioms are the standard axioms used in the literature. In this paper, we introduce a new axiom and use it while comparing mechanisms based on their costs.

In matching μ, let $\Upsilon(\mu)$ be the set of existing tenants assigned to the house where they are currently living, $\Upsilon(\mu) = \{i \in I_E | \mu(i) = h_i\}$. A matching μ is **less costly** than μ' if $|\Upsilon(\mu)| > |\Upsilon(\mu')|$. A mechanism φ is **less costly** than mechanism Φ if in any problem the matching selected by Φ is not **less costly** than the matching selected by φ and there exists a problem in which the matching selected by φ is less costly than the one selected by Φ.

3.3 Competing Mechanisms

In this section we describe two individually rational mechanisms that are proposed in the literature and the widely used RSDwSR mechanism. In all three mechanisms, each existing tenant decides whether she will stay out of the assignment process and keep her current house or participate and apply for a new house. Without loss of generality, we consider all agents in I_E as the existing tenants who decide to enter the assignment process. Therefore, each $h \in H_O$ is owned by an existing tenant who enters the assignment process.

Given a house allocation problem $[I_E, I_N, H_O, H_V, P, f]$, the outcome of DA mechanism is obtained by applying it to the associated problem $[I, H, P, \succ]$.[12]

Deferred Acceptance Mechanism:

Step 1: Each agent applies to his first choice house. Each house h tentatively accepts the agent having the highest priority in $\succ_h$ among the agents who applied to it. The rest are rejected.

In general:

Step k: Each agent applies to the best house that has not rejected him. Each house h tentatively accepts the agent having the highest priority in $\succ_h$ among the agents who applied to it. The rest are rejected.

The algorithm terminates when no agent is rejected any more.

For the TTC mechanism we also work on the associated problem $[I, H, P, \succ]$. The outcome of the TTC mechanism can be found by the following algorithm.

Top Trading Cycles Mechanism:

Step 1: Each agent points to his first choice house in his preference list. Each house $h \in H$ points to the agent who has the highest priority in $\succ_h$. The null house points to the agents pointing to it. There always exists at least one

[12] In Guillen and Kesten (2012) the associated problem is considered as a school choice problem, which was introduced by Abdulkadiroğlu and Sönmez (2003).

cycle due to the finite number of houses and agents. Assign the agents in the cycle to the house they point to and remove them with their assignments.

In general:

Step k: Each remaining agent points to the best house among the remaining ones. Each house h points to the agent who has the highest priority in $\succ_h$ among the remaining ones. The null house points to the agents pointing to it. There exists at least one cycle. Assign the agents in the cycle to the house that they point to and remove them with their assignments.

The algorithm terminates when all agents are removed.

For the RSDwSR mechanism we work on the house allocation problem $[I_E, I_N, H_O, H_V, P, f]$. The outcome of the RSDwSR mechanism can be found by the following algorithm.

Random Serial Dictatorship with Squatting Rights:

Step 1: Agent $f(1)$ is assigned to his top choice house. Remove $f(1)$. If $f(1)$ is assigned to a house in H, then remove his assignment.

Step 2: Agent $f(2)$ is assigned to his top choice house among the remaining ones. Remove $f(2)$. If $f(2)$ is assigned to a house in H, then remove his assignment.

In general:

Step k: Agent $f(k)$ is assigned to his top choice house among the remaining ones. Remove $f(k)$. If $f(k)$ is assigned to a house in H, then remove his assignment.

The algorithm terminates when all agents are removed.

Both the DA and TTC mechanisms are strategy-proof and individually rational. The DA mechanism is fair but not Pareto efficient. On the other hand, TTC is Pareto efficient but not fair. Kesten (2006) shows that if DA fails to be Pareto efficient for a problem TTC fails to be fair. However, there are some problems in which TTC fails to be fair but the DA mechanism selects a Pareto efficient matching. The RSDwSR mechanism does not satisfy individual rationality (Abdulkadiroğlu and Sönmez, 1999). It is strategy proof and Pareto efficient if we only consider the participants. However, it fails to be Pareto efficient within the set of all agents including both nonparticipants and participants.

All these comparisons are done using the axioms that are related only to the agents. If the housing office (central authority) is benevolent, then these comparisons will help the housing office decide which individually rational and strategy-proof mechanism to adopt. Since none of the axioms used in the comparisons considers the preference of the housing office, these comparisons do not help the housing office if it is not benevolent.

3.4 Results

In this section we first compare the two individually rational and strategy-proof mechanisms based on the number of existing tenants assigned to a different house under each mechanism. Each movement of an existing tenant is considered a cost to the housing office. Then we redefine Pareto efficiency by including the welfare of the housing office and do our analysis based on this new definition of Pareto efficiency.

Before comparing the DA and TTC mechanisms, we first answer two natural questions: (1) Is RSDwSR mechanism less costly than both the DA and TTC mechanisms? (2) Is it less costly to rank all the existing tenants after all the newcomers in f under both the DA and TTC mechanisms?

If the answer to the first question is "Yes" then it will be difficult to influence the housing office to replace RSDwST with the DA or TTC mechanism. We answer the first question by showing that RSDwSR may not be the best solution for reducing the costs to housing offices. In Example 8, we illustrate that in some problems the outcome selected by TTC and DA mechanisms can be less costly than the outcome selected by the RSDwSR mechanism.

Example 8 *There are three students* $I = \{i, j, k\}$ *and three houses* $H = \{h_1, h_2, h_3\}$. *Agent* k *is currently living in* h_3 *and he is an expected utility maximizer. Let* $u(h)$ *be the utility of* k *from being assigned to house* h *and* $u(h_1) = 13$, $u(h_2) = 0$ *and* $u(h_3) = 5$. *The preference profile is given as:* $h_1 P_i h_3 P_i h_2$, $h_1 P_j h_3 P_j h_2$, $h_1 P_k h_3 P_k h_2$.

Based on the ordering f *we get the following outcomes under serial dictatorship.*

Ordering	*Assignment of Agent k*	*Utility*
$i-j-k$	h_2	*0*
$i-k-j$	h_3	*5*
$j-i-k$	h_2	*0*
$j-k-i$	h_3	*5*
$k-i-j$	h_1	*13*
$k-j-i$	h_1	*13*

Suppose the distribution of orderings is uniform. The expected utility of agent k from participating in the assignment process is 6, which is greater than the utility of keeping her current house. Therefore student k will participate. If the selected ordering is $i-j-k$ then k will be assigned to h_2 by RSDwSR. However, for this ordering both TTC and DA mechanisms assign k to his own house. Moreover, in none of the orderings the outcome of the RSDwSR is less costly than the outcome of TTC or DA mechanisms.

Example 8 does not show that either DA or TTC is less costly than RSDwSR. One can show that with a different distribution of orderings or different utilities student k will decide not to participate, whereas the DA and TTC mechanism may assign him to another house. For instance, take utilities as: $u'(h_1) = 7$, $u'(h_2) = 0$ and $u'(h_3) = 5$. Then the expected utility of agent k from participating is 4 which is less than the utility of keeping her current house. Therefore k will not participate. On the other hand, if the selected ordering is $k-i-j$ both mechanisms will assign her to h_1.

In Example 8, we assume that existing tenants give their participation decision before seeing the exact ordering f. This assumption is consistent with Abdulkadiroğlu and Sönmez (1999). It might be the case that existing tenants give their participation decision after the realization of f. Even if this is the case, not knowing the exact preferences of the other students also leads to uncertainty over the outcomes of the assignment procedure for the existing tenants. In this case we can come up with the same expected utility by using a belief structure of the existing tenant about the preferences of the other applicants.

For the second question, since ranking all the existing tenants after all the newcomers gives them lower priorities, the immediate answer that comes to mind is "Yes". But this immediate answer is not true. In the following example we show that ranking the existing tenants after all the newcomers in f under the DA and TTC mechanisms does not always yield a less costly outcome for the housing office.

Example 9 *We first focus on the DA mechanism. There are four houses, $H = \{h_1, h_2, h_3, h_4\}$ and four agents $I = \{i_1, i_2, i_3, i_4\}$. Agent i_j is currently living in h_j where $j \in \{1, 2, 3\}$. That is, i_1, i_2 and i_3 are the existing tenants. Let the preferences of the agents be: $h_2 P_{i_1} h_1 P_{i_1} h_3 P_{i_1} h_4$, $h_1 P_{i_2} h_2 P_{i_2} h_3 P_{i_2} h_4$, $h_4 P_{i_3} h_3 P_{i_3} h_2 P_{i_3} h_1$ and $h_4 P_{i_4} h_1 P_{i_4} h_3 P_{i_4} h_2$.*

First consider the outcome of the DA mechanism for the following order: $f : i_3 - i_4 - i_1 - i_2$. The DA mechanism selects the following matching $\mu(i_1) = h_1$, $\mu(i_2) = h_2$, $\mu(i_3) = h_4$ and $\mu(i_4) = h_3$. The number of existing tenants keeping their own house is two.

Now consider the outcome of the DA mechanism for the following order: $f' : i_4 - i_3 - i_1 - i_2$. *The DA mechanism selects the following matching:* $\mu'(i_1) = h_2$, $\mu'(i_2) = h_1$, $\mu'(i_3) = h_3$ *and* $\mu(i_4) = h_4$. *The number of existing tenants keeping their own house is 1.*

Now we focus on the TTC mechanism. We consider the same example above with a slight change in the preferences. Consider the following preference profile: $h_2 P'_{i_1} h_1 P'_{i_1} h_3 P'_{i_1} h_4$, $h_3 P'_{i_2} h_2 P'_{i_2} h_1 P'_{i_2} h_4$, $h_4 P'_{i_3} h_1 P'_{i_3} h_2 P'_{i_3} h_3$ *and* $h_4 P'_{i_4} h_3 P'_{i_4} h_1 P'_{i_4} h_2$.

Consider the outcome of the TTC mechanism for the following order: $f : i_3 - i_4 - i_1 - i_2$. *The TTC mechanism selects the following matching:* $\pi(i_1) = h_1$, $\pi(i_2) = h_2$, $\pi(i_3) = h_4$ *and* $\pi(i_4) = h_3$. *The number of existing tenants keeping their own house is 2.*

Now consider the outcome of the TTC mechanism for the following order: $f' : i_4 - i_3 - i_1 - i_2$. *The TTC mechanism selects the following matching:* $\pi'(i_1) = h_2$, $\pi'(i_2) = h_3$, $\pi'(i_3) = h_1$ *and* $\pi'(i_4) = h_4$. *The number of existing tenants keeping their own house is 1.*

When we compare both mechanisms in a general setting, TTC (DA) is not less costly than DA (TTC). However, under some restrictions DA is less costly than TTC. In particular, if the number of existing tenants is one, then the DA mechanism is less costly than the TTC mechanism. We formally state this result in Proposition 18.[13]

[13]When the number of existing tenants is zero, then DA and TTC will select the same

Proposition 18 *If $|I_E| = 1$, then there does not exist a problem in which the matching selected by TTC is less costly than the one selected by DA, but there exist problems in which the matching selected by DA is less costly than the one selected by TTC.*

Denote the matching selected by TTC and DA by π and μ, respectively. We prove the first part by contradiction. Suppose the statement is not true. Then there exists a problem in which the matching selected by TTC is less costly than the matching selected by DA. This is possible if and only if $i \in I_E$ is assigned to h_i under TTC and assigned to another house under DA. Given that μ is individually rational, i cannot be assigned to a worse house than h_i. That is, $\mu(i) P_i \pi(i)$.

We will consider a variant of the TTC mechanism in which only one cycle is removed in each step. Let $\mu(i)$ be removed in step k. Then the priority order of student i cannot be less than or equal to k. Otherwise, i will be placed to $\mu(i)$ or a better house. Then, $\{f(1), f(2), ..., f(k)\} \subseteq I_N$. Student $f(1)$ is assigned to his top choice in π. Student $f(2)$ is assigned to his top choice in step 2 among the remaining houses. It continues like this until step k. All these houses are new houses.

Now consider the sequential version of the DA mechanism (McVitie and Wilson, 1971). Students will apply based on their rank in f. $f(1)$ will apply to $\pi(f(1))$ and is tentatively kept. Then it is student $f(2)$'s turn. If he ranks

matching for every problem (see Appendix C).

$\pi(f(1))$ at the top when he applies to that house he will be rejected. Since $f(2)$ prefers $\pi(f(2))$ most among $H \setminus \{\pi(f(1))\}$ then he will be tentatively accepted when he applies to $\pi(f(2))$ next. This will be the same for all students in $\{f(1), f(2), ..., f(k)\}$. Therefore, i cannot be assigned to $\mu(i)$ under the DA mechanism. This leads to a contradiction.

We prove the second part by example. There are 3 houses $H = \{h_1, h_2, h_{i_3}\}$ and 3 students $I = \{i_1, i_2, i_3\}$. Here i_3 is an existing student currently occupying h_{i_3}. Let $f(1) = i_1$, $f(2) = i_2$ and $f(3) = i_3$. The preference profile is given by: $h_{i_3} P_{i_1} h_1 P_{i_1} h_2$, $h_1 P_{i_2} h_2 P_{i_2} h_{i_3}$ and $h_1 P_{i_3} h_{i_3} P_{i_3} h_2$.

In this problem TTC will select the following matching: $\pi(i_1) = h_{i_3}$, $\pi(i_2) = h_2$ and $\pi(i_3) = h_1$. On the other hand the DA mechanism will select the following matching: $\mu(i_1) = h_1$, $\mu(i_2) = h_2$ and $\mu(i_3) = h_{i_3}$. The existing tenant is assigned to a different house in π but stays in his house in μ.

When $|I_E| > 1$ there is no dominance between DA and TTC in terms of cost efficiency (being less costly). We illustrate this situation in Example 10.

Example 10 *There are three students $I = \{i, j, k\}$ and three houses $H = \{h_1, h_2, h_3\}$. Student i and j are existing tenants occupying h_1 and h_2, respectively. Let $f(1) = k$, $f(2) = j$ and $f(3) = i$. The preference profile is given by: $h_3 P_i h_2 P_i h_1$, $h_3 P_j h_2 P_j h_1$ and $h_1 P_k h_2 P_k h_3$. DA will select the following matching: $\mu(i) = h_2$, $\mu(j) = h_3$ and $\mu(k) = h_1$. On the other hand, TTC will select the following matching: $\pi(i) = h_3$, $\pi(j) = h_2$ and $\pi(k) = h_1$. In μ all existing* *tenants change their places. In π only one existing tenant changes his house.*

Now consider a new preference profile: $h_2 P'_i h_1 P'_i h_3$, $h_1 P'_j h_2 P'_j h_3$ and $h_1 P'_k h_2 P'_k h_3$. In this problem, DA will select the following matching: $\mu'(i) = h_1$, $\mu'(j) = h_2$ and $\mu'(k) = h_3$. TTC will select the following matching: $\pi'(i) = h_2$, $\pi'(j) = h_1$ and $\pi'(k) = h_3$. In π' all existing tenants change their places. In μ' all existing tenants stay in their houses.

In Example 10 we look only at the case in which $|I_E| = 2$. We can easily modify the example by adding more existing tenants who prefer their own houses. Under both mechanisms, the existing tenants who prefer their own houses will be assigned to their own houses. That is, adding other agents will not change the assignments of the three agents considered in Example 10. Therefore, we have the same result for all cases where $|I_E| \geq 2$.

When we consider the on-campus housing assignment system, it is quite true that the number of existing tenants applying to be reassigned is greater than one. Therefore, Proposition 18 does not give enough support to the adoption of DA instead of TTC. Although there is no dominance between the DA and TTC mechanisms in terms of cost efficiency when $|I_E| > 1$, in the following proposition we show that the DA mechanism has better features than TTC in terms of cost efficiency.

Proposition 19 *If $|I_E| \geq 2$ and at least one of the existing tenants is not in $\{f(1), f(2)\}$, then for any set of agents, houses, and priority ordering there always exists a preference profile P such that the matching selected by DA is* *less costly than the matching selected by TTC. But the converse is not true: There exists a set of agents, houses, and priority where there does not exist a preference profile P such that the matching selected by TTC is less costly than the matching selected by DA.*

We first show the first part of the statement. Let $i \in I_E$ and $i \notin \{f(1), f(2)\}$. Consider the following problem. Except for i, each $j \in I_E$ such that $j \notin \{f(1), f(2)\}$ prefers h_j most. There are 4 possible cases.

Case 1: $\{f(1), f(2)\} \subseteq I_E$. Student i and $f(2)$ prefer house $h_{f(1)}$ most and their own houses second. Student $f(1)$ prefers h_i most and his own house second. DA will assign all existing agents to their current houses. On the other hand $f(1)$ and i will swap their houses in the matching selected by TTC.

Case 2: $f(1) \in I_E$ and $f(2) \in I_N$. Student i and $f(2)$ prefer a new house $h \in H_N$ most. Student i prefers his own house second. Student $f(2)$ prefers $h_{f(1)}$ second. Student $f(1)$ prefers h_i most and his own house second. DA will assign all existing agents to their current houses. On the other hand $f(1)$ gets h_i and i gets h in the matching selected by TTC.

Case 3: $f(1) \in I_N$ and $f(2) \in I_E$. Student i and $f(2)$ prefer a new house $h \in H_N$ most. Student i and $f(2)$ prefer their own houses second. Student $f(1)$ prefers h_i most and h second. DA will assign all existing agents to their current houses. On the other hand i is assigned to h in the matching selected by TTC.

Case 4: $\{f(1), f(2)\} \subseteq I_N$. Student i and $f(2)$ prefer a new house

$h \in H_N$ most. Student i prefers his own house second. Student $f(2)$ prefers $h' \in H_N$ second. Student $f(1)$ prefers h_i most and $h' \in H_N$ second. DA will assign all existing agents to their current houses. On the other hand i is assigned to h in the matching selected by TTC.

Now, we prove the second part of Proposition 19. There are two existing students $I_E = \{i, k\}$ and one newcomer $I_N = \{j\}$. The priority order is $f(1) = i$, $f(2) = j$ and $f(3) = k$. Let h_n be the new house and h_i and h_k be current houses of i and k, respectively. There are 216 possible combinations of preferences.

In all preference profiles in which i ranks h_n or h_i at the top, DA and TTC will select the same matching. This corresponds to 144 preference profiles.

In all preference profiles in which i and k prefer h_k most, DA and TTC will select the same matching. This corresponds to 24 preference profiles.

In all preference profiles in which i prefers h_k and k prefers h_i or h_n the most, i and k will be assigned to their top choice in TTC. Therefore both existing tenants will move to another house. This corresponds to 48 preference profiles. Note that TTC cannot be less costly than DA in this case because all existing tenants are assigned to the other houses.

If i prefers h_k the most and h_i second, j and k prefers h_i the most and k prefers h_k second then DA will assign each existing tenant to his current house.

According to Proposition 19, for any priority structure it is always possible that DA selects a less costly assignment than the outcome of TTC when we have more than one existing tenant.[14]

In Proposition 19 we restrict one of the existing tenant to not be in $\{f(1), f(2)\}$ because when all existing tenants belong to that set the DA and TTC mechanisms select the same outcome (see Appendix C).

We cannot consider the housing office to be totally selfish. It may also take into account the welfare of the agents. Then we should consider the performance of these two mechanisms in other aspects. In Proposition 20, we show that if the outcome of TTC is less costly than the outcome of DA, then the outcome of TTC fails to be fair. However, in some problems the outcome of the DA mechanism can be Pareto efficient and less costly than the outcome of the TTC mechanism at the same time.

Proposition 20 *Whenever the matching selected by TTC is less costly than the matching selected by DA, then the matching selected by TTC is not fair. But there exists a problem in which the matching selected by DA is Pareto efficient and less costly than the outcome of the TTC mechanism.*

Let π and μ be the matchings selected by TTC and DA, respectively. If the matching selected by TTC is less costly than the matching selected by DA, then $\pi \neq \mu$. The outcome of DA Pareto dominates all the other fair

[14] This result is also true for the case where $|I_E| = 1$.

matchings and the outcome of TTC is Pareto efficient; therefore π cannot be fair.

For the second part consider the following example. There are three students $I = \{i, j, k\}$ and three houses $H = \{h_1, h_2, h_3\}$. Student k is currently living in h_3. The priority order is $f(1) = i$, $f(2) = j$ and $f(3) = k$. The preference profile is given as: $h_3 P_i h_1 P_i h_2$, $h_2 P_j h_3 P_j h_1$, $h_2 P_k h_3 P_k h_1$. TTC will select the following matching: $\pi(i) = h_3$, $\pi(j) = h_1$ and $\pi(k) = h_2$. On the other hand, DA will select the following matching: $\mu(i) = h_1$, $\mu(j) = h_2$ and $\mu(k) = h_3$. μ is Pareto efficient and less costly than π.

So far, we have considered only the welfare of the agents in our analysis. Now, we include the welfare of the the housing office and and analyze the performance of the two mechanisms in this new environment. By including the welfare of the housing office in our analysis we introduce a new class of problem that we call a **1.5-sided house allocation problem with existing agents** (1.5-sided HAPwET). In particular, we change the HAPwET by adding the housing office to the set of agents and the preferences of the housing office into the preference profile. We define 1.5-sided HAPwET as a 6-tuple $[I_E, I_N, o, H_O, H_V, (P_x)_{x \in \tilde{I}}, f]$ where o represents the housing office and $\tilde{I} = I_E \cup I_N \cup o$.

We assume that the housing office strictly prefers matching μ to matching ν if the number of existing tenants assigned to their own houses is greater under μ and the housing office is indifferent between two allocations if the number of existing tenants assigned to their own houses is the same in both matchings. Formally, (1)$\mu P_o v$ if $\Upsilon(\mu) > \Upsilon(\nu)$, (2) $\mu R_o \nu$ and $v R_o \mu$ if $\Upsilon(\mu) = \Upsilon(\nu)$. In this new class of problems we only change the definition of the Pareto-efficiency axiom that we defined for the one-sided problems. In particular, we change the definition of Pareto efficiency by including the housing office in the welfare analysis in addition to the agents: A matching $\mu \in \mathcal{M}$ **Pareto dominates** another matching $\nu \in \mathcal{M}$ if $\mu(i)\ R_i\ \nu(i)$ for all $i \in I \cup o$ and $\mu(j)\ P_j\ \nu(j)$ for some $j \in I \cup o$. A matching is **Pareto efficient** if it is not Pareto dominated by any other matching. In the rest of the paper, we use this definition of Pareto efficiency which includes the welfare of the housing office.

In 1.5-sided HAPwET, we run the TTC and DA mechanisms without taking the preferences of the housing office into account.[15] Since we only change the definition of Pareto efficiency and keep everything else the same and run both mechanisms as in the one-sided HAPwET, DA satisfies individual rationality, fairness, and strategy-proofness in 1.5-sided HAPwET. Similarly, TTC is individually rational and strategy-proof. Moreover, TTC still satisfies Pareto efficiency. We formally show this result in the following proposition.

Proposition 21 *TTC is Pareto efficient in the 1.5-sided HAPwET.*

Suppose not. There exists another matching, ν that Pareto dominates the outcome of the TTC mechanism, π. Then, there exists at least one agent

[15] On the other hand, in two-sided problems the preferences of both sides are taken into account by the two-sided versions of TTC and DA (see Dur and Ünver, 2012; Gale and Shapley, 1962).

whose allocations are different in π and ν and he prefers ν to π. Otherwise, ν cannot Pareto dominate π. Let i be the agent preferring $\nu(i)$ to $\pi(i)$. Therefore i is not assigned to his top choice in π.

Denote the set of agents removed in step k of TTC with $C(k)$. The agents removed in the first step of TTC are assigned to their top choices. Therefore, i cannot be in this group. Now consider the agents removed in the second step of TTC. Suppose $i \in C(2)$. Then $\nu(i)$ should be i's top choice. The only reason why i is not assigned to his top choice is that another agent in $C(1)$ is assigned to that house. That is, one of the agents in $C(1)$ is assigned to $\nu(i)$ in π. Denote this agent with j. If i is assigned to $\nu(i)$ then j is assigned to another house in ν. Given that $\pi(j) = \nu(i)$ is the top choice of j, agent j prefers π to ν. This is a contradiction. We can continue with the other cycles similarly and show that we cannot find a matching that Pareto dominates π.

Proposition 21 holds if only students have strict preferences over the houses. This is also true for the one-sided HAPwET. That is, if agents have weak preferences over the houses then TTC fails to be Pareto efficient in the one-sided problem. Note that Proposition 21 holds independent of the preference profile of the housing office.

In the two-sided allocation problems, DA satisfies Pareto efficiency. One might wonder whether we can get this result in the 1.5-sided HAPwET. In the following theorem we show that when the housing office is included in the welfare analysis, any fair matching is also Pareto efficient.

Theorem 12 *In 1.5-sided HAPwET, if matching μ is fair then it is also Pareto efficient.*

Suppose there exists another matching ν that Pareto dominates μ. Then there exists a set of agents $J \subseteq I \cup o$ strictly preferring ν to μ and all the other agents are indifferent between ν and μ. Since agents preferences are strict, any agent who is assigned to different houses in ν and μ should prefer ν. If $o \in J$, then there exists an existing tenant i such that $\mu(i)$ is not his own house whereas $\nu(i)$ is. Moreover, $\nu(i) P_i \mu(i)$. However, this violates the fairness of matching μ because he has the highest priority for $\nu(i)$ and is assigned to a worse house in μ. Therefore, o is indifferent between ν and μ, and $J \subseteq I$. Suppose J is a singleton set. Then, $j \in J$ should be assigned to an empty house in μ. However, this contradicts the fairness of μ. Moreover, any agent in J cannot be assigned to a house in ν that is empty under μ. Therefore, J cannot be singleton and for each agent $j \in J$ there exists another agent $j' \in J$ such that $\nu(j) = \mu(j')$. Then, we can find a cycle $\tilde{J} \subseteq J = \{j_1, j_2, ..., j_n\}$ where $\nu(j_x) = \mu(j_{x-1})$ for $x > 1$ and $\nu(j_1) = \mu(j_n)$. Since μ is fair, $j \in \tilde{J}$ has higher priority for $\mu(j)$ than $\nu^{-1}(\mu(j))$. We claim that at least one agent in $\tilde{J}$ is assigned to his own house in μ. Suppose not. Then the only way agent j_x has a higher priority for $\mu(j_x)$ than j_{x+1} is if he is ranked in a better position in the ordering f. Then, $f^{-1}(j_1) < f^{-1}(j_2)$, $f^{-1}(j_2) < f^{-1}(j_3)$,..., $f^{-1}(j_{n-1}) < f^{-1}(j_n)$ and $f^{-1}(j_n) < f^{-1}(j_1)$. This can be written as: $f^{-1}(j_1) < f^{-1}(j_2) < f^{-1}(j_3) < ... < f^{-1}(j_n) < f^{-1}(j_1)$. This is a contradiction. Moreover, due to fairness there cannot be an existing tenant assigned to his own house in ν but not in μ.

Given that the DA is fair as a consequence of Theorem 12 it is also Pareto efficient.

Corollary 4 *In the 1.5-sided HAPwET, the DA mechanism is Pareto efficient.*

Even if we include the housing office in the welfare analysis, it is not a strategic agent. That is, it cannot determine which ordering f will be selected. Only agents are strategic and they can manipulate their preferences. Alcalde and Barbera (1994) show that DA is the unique strategy-proof and fair mechanism when the other side of the market is not strategic. As a result, DA is the unique fair, strategy-proof, and Pareto efficient mechanism.

Theorem 12 shows us that any fair mechanism also satisfies Pareto efficiency. Then, do agents and the housing office have a favorite fair mechanism? Before answering this question we give some definitions. We say that a fair matching μ is the **best fair matching** for agents if every agent likes it at least as well as any other fair matchings, and μ is the **worst fair matching** for agents if every agent likes any other fair matching at least as well as μ. Similarly, a fair matching μ is the **best fair matching** for the housing office if the housing office likes it at least as well as any other fair matching and μ is the **worst fair matching** for the housing office if the housing office likes any other fair matching at least as well as μ. In the marriage problem, introduced by

Gale and Shapley (1962), we can find best fair matchings for men and women. This is also true for the house allocation problem with existing tenants. That is, if we consider agents as one side of the market and the housing office as the other side, then there always exist best and worst fair matchings for both the housing office and the agents. Moreover, the best fair matching for the agents (housing office) is the worst fair matching for the housing office (agents). We show these results in Proposition 22.

Proposition 22 *There always exist best and worst fair matchings for agents and the housing office. Moreover, the best (worst) fair matching for the agents is the worst (best) fair matching for the housing office.*

Consider the house allocation problem $[I_E, I_N, H_O, H_V, P, f]$. Then construct the associated problem $[I, H, P, \succ]$ as described in Section 3.2. This is a school choice problem in which every school has only one available seat. In the school choice problem, school proposing DA mechanism gives the worst fair matching for the agents and agent proposing DA mechanism gives the best fair matching for the agents. For the polarization we refer to the proof of Theorem 12. Let μ_B and μ_W be the best and worst fair matchings for agents, respectively. If there is a unique fair matching, then the best and worst fair matchings are the same for both sides. If there is more than one fair matching we can Pareto rank all the other fair matchings with μ_B and μ_W for the agents (Roth and Sotomayor, 1990). If $\mu \neq \mu_B$ then μ_B is weakly preferred over μ by agents. As we show in the proof of Theorem 12, μ_B is more costly than μ

and less preferred than μ by the housing office. Therefore μ_B is the costliest matching. If $\mu \neq \mu_W$, then μ is weakly preferred to μ_W by agents. As we show in the proof of Theorem 12, μ_W is less costly than μ and preferred to μ by the housing office. Therefore μ_W is the most preferred fair matching for the housing office.

In Proposition 22, we show that there always exist best and worst stable matchings for each side of the market. Then a natural question is whether we can find the best Pareto efficient matching for both the agents and the housing office. Since one side of the market consists of only one agent, we can always rank the Pareto efficient matchings for the housing office. Given that the housing office has weak preferences over the matchings it can be indifferent between some Pareto efficient matchings and strictly prefer them to other Pareto efficient matchings. Thus, there is a set of best Pareto efficient matchings for the housing office. For the agents' side, there may not exist a Pareto efficient matching that is preferred to any other Pareto efficient matching by all agents. It is easy to see that the best Pareto efficient matching for the agents exists if and only if the most popular house of each agent is different. In that case, TTC and DA will select the same matchings. On the other hand, DA and TTC do not have the same success in selecting a matching from the set of best Pareto efficient matchings for the housing office. We show this difference in Proposition 23.

Proposition 23 *If TTC selects a matching from the set of best Pareto efficient matchings for the housing office then DA selects the same matching. On* *the other hand, there exists a problem in which DA selects a matching from the set of best Pareto efficient matchings but TTC does not.*

We start with the first statement. We first claim that the set of best Pareto efficient matchings for the housing office consists of Pareto efficient matchings in which all existing tenants are assigned to their own houses. To prove this claim we need to show that there always exists a Pareto efficient matching in which all existing tenants are assigned to his current house. Consider the following mechanism: In all problems each existing tenant is assigned to their current houses. Then select a random order for the newcomers and run the serial dictatorship mechanism based on the random draw in order to assign newcomers to the vacant houses. Any Pareto improving trade between agents includes at least one existing tenant and will make the housing office worse off. Therefore this mechanism always selects a Pareto efficient matching.

Denote the outcome of TTC with π. If TTC assigns each existing tenant to his own house then the outcome should be fair. Suppose not. First note that there is no priority violation for the occupied houses. If there is a priority violation then there exist two agents i and j such that $f^{-1}(i) < f^{-1}(j)$ and $\pi(j)P_i\pi(i)$. Here j is a newcomer and in the TTC mechanism a newcomer agent will be assigned to a house after all the agents with higher priority in f are removed. That is, when i was removed $\pi(j)$ was available. This is a contradiction and π is fair. When we only consider the welfare of the agents, TTC selects an undominated matching in every problem. On the other hand, the outcome of the DA mechanism Pareto dominates all other fair matchings. Therefore, DA also selects π.

For the second part, consider the second part of Example 10. In that example DA selects a matching from the set of best Pareto efficient matchings but TTC fails.

One may think that including the housing office in the welfare analysis is equivalent to including all the houses in the welfare analysis.[16] Once both the agents and houses are considered in the welfare analysis, the house allocation problem with existing tenants turns into the marriage problem introduced by Gale and Shapley (1962). In the marriage problem, fairness (stability) implies Pareto efficiency. In Theorem 12, we show that the same result holds in our context. However, we later show (Theorem 13) that Theorem 12 does not hold if we make more general assumptions about the priority structure. Thus, including the housing office in the welfare analysis is not equivalent to including all the houses in the welfare analysis.

Until now, we have focused on the case in which each house ranks agents other than its existing tenant based on the same ordering f. Now we extend the model by allowing each house to use different orderings. It is worth mentioning that our extension does not preclude the possibility that two different houses have the same ordering. For instance, different dorms on a campus might give higher priority to the students who lived in that dorm in the previous

[16] In this case, priorities of the houses can be considered as preferences.

year or for some dorms honor students might have higher priority. For each house $h \in H$, let f_h be the ordering over the set of agents. Let $h \in H_V$ and $h_k \in H_O$. Given the ordering f_h, $f_{h'}$ and the ownership profile we construct the priority order for house $h \in H_V$, $\succ_h$, as $i \succ_h j$ if $f^{-1}(i) < f^{-1}(j)$ and for house $h_k \in H_O$, $k \succ_{h_k} j$ for all $j \in I \setminus k$ and for each $l, m \in I \setminus k$ $l \succ_{h_k} m$ if $f^{-1}(l) < f^{-1}(m)$. Under this extension, Theorem **??** does not hold. In Theorem **??**, we show that there exists a problem in which all fair matchings fail to be Pareto efficient.

Theorem 13 *If different orderings are used for houses then there may exist a problem in which all fair matchings are Pareto dominated.*

There are 3 houses $H = \{h_1, h_2, h_3\}$ and 3 students $I = \{i_1, i_2, i_3\}$. All agents are newcomers and all houses are vacant. Let $f_{h_1} = i_1 - i_2 - i_3$, $f_{h_2} = i_3 - i_1 - i_2$ and $f_{h_3} = i_2 - i_1 - i_3$. The preference profile is given by: $h_2 P_{i_1} h_1 P_{i_1} h_3$, $h_1 P_{i_2} h_3 P_{i_2} h_2$ and $h_1 P_{i_3} h_2 P_{i_3} h_3$. There is a unique fair matching: $\mu(i_1) = h_1$, $\mu(i_2) = h_3$ and $\mu(i_3) = h_2$. The unique fair matching is Pareto dominated by the following matching: $\mu'(i_1) = h_2$, $\mu'(i_2) = h_3$ and $\mu'(i_3) = h_1$.

In Theorem 13, we also show that there does not exist a fair and Pareto efficient mechanism by showing that the unique fair matching is not Pareto efficient.

In the 1.5-sided problem where all houses use the same ordering f, the priority structure totally reflects the preferences of the housing office over the matchings. Therefore, an allocation that is Pareto efficient based on the preferences of the agents and preferences (priority orders) of houses also satisfies Pareto efficiency in the 1.5-sided problem. However, when houses use a different ordering the priority structure does not reflect the preferences of the housing office. This is the main reason why Theorem **??** does not hold when houses rank agents based on different orderings.

In the one sided problems, if the priority orderings of objects are heterogeneous[17] then fairness and Pareto efficiency are incompatible (Balinski and Sönmez, 1999). If there exists a fair and Pareto efficient matching, it is unique and it is the outcome of the DA mechanism. When we increase the level of heterogeneity in the priority orders by using different orderings for house, there may exist more than one Pareto efficient and fair matching. However, the DA mechanism may fail to select one of the Pareto efficient and fair matchings. We illustrate this situation in Example 11.

Example 11 *There are 7 agents, $I = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7\}$, and 7 houses, $H = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7\}$. Agent i_1, i_2 and i_6 are existing tenants in houses h_1, h_2, and h_6, respectively. The preference profile is given by: $h_4 P_{i_1} h_3 P_{i_1} h_1$, $h_3 P_{i_2} h_4 P_{i_2} h_2$, $h_1 P_{i_3} h_3$, $h_2 P_{i_4} h_4$, $h_3 P_{i_5} h_5$, $h_7 P_{i_6} h_6$ and $h_6 P_{i_7} h_7$. The orderings for each house are given as: $f_{h_1} = f_{h_2} = i_1 - i_2 - i_3 - i_4 - i_5 - i_6 - i_7$, $f_{h_3} = f_{h_6} = i_3 - i_1 - i_5 - i_2 - i_4 - i_6 - i_7$ and $f_{h_4} = f_{h_5} = f_{h_7} = i_4 - i_2 - i_1 - i_5 - i_3 - i_7 - i_6$*

.

[17]By heterogeneity we mean that objects are not ordering the agent in the same order.

In this example the following fair matchings are Pareto efficient: (1) $\mu(i_1) = h_1$, $\mu(i_2) = h_2$, $\mu(i_3) = h_3$, $\mu(i_4) = h_4$, $\mu(i_5) = h_5$, $\mu(i_6) = h_6$, $\mu(i_7) = h_7$, *(2)* $\mu'(i_1) = h_1$, $\mu'(i_2) = h_2$, $\mu'(i_3) = h_3$, $\mu'(i_4) = h_4$, $\mu'(i_5) = h_5$, $\mu'(i_6) = h_7$, $\mu'(i_7) = h_6$. *However, the outcome selected by DA fails to be Pareto efficient:* $\mu''(i_1) = h_3$, $\mu''(i_2) = h_4$, $\mu''(i_3) = h_1$, $\mu''(i_4) = h_2$, $\mu''(i_5) = h_5$, $\mu''(i_6) = h_6$, $\mu''(i_7) = h_7$. *It is Pareto dominated by the following matching:* $\tilde{\mu}(i_1) = h_4$, $\tilde{\mu}(i_2) = h_3$, $\tilde{\mu}(i_3) = h_1$, $\tilde{\mu}(i_4) = h_2$, $\tilde{\mu}(i_5) = h_5$, $\tilde{\mu}(i_6) = h_6$, $\tilde{\mu}(i_7) = h_7$.

Theorem 13 and Example 11 explicitly show that although the 1.5-sided problem inherits properties of both the one-sided and two-sided problems it has its own characteristic features and is different from the other two problems.

When we allow different orders for houses, in a fair matching there may exist welfare improving trades between agents that do not make the housing office worse off. For fairness to imply Pareto efficiency, a relation between the priority structure and the preference of the housing office is required. In the following theorem, we give the condition on the preference of the housing office guaranteeing that every fair matching satisfy Pareto efficiency.

Theorem 14 *Every fair matching is Pareto efficient if whenever there exist a set of students* $\tilde{I} = \{i_1, i_2, ..., i_n\}$ *and a set of houses* $\tilde{H} = \{h_1, h_2, ..., h_n\}$ *such that* $i_n \succ_{h_n} i_{n-1} \succ_{h_{n-1}} i_{n-2} \succ_{h_{n-2}} ...i_1 \succ_{h_1} i_n$ *then the housing office strictly prefers matching* μ *to* ν *where* $\mu(i_k) = \nu(i_{k-1}) = h_k$ *for all* $k \in \{2, 3, ..., n\}$ *and* $\mu(i_1) = \nu(i_n) = h_1$.

Let π be a fair matching. If there does not exist another matching π' that is weakly preferred by all agents and strictly preferred at least by some agents to π, then π is Pareto efficient (see Proposition 21). Otherwise, there exists a set of agents $\tilde{I}$ who strictly prefer their assignment in π' to π. We claim that for each $i \in \tilde{I}$ there exists another $j \in \tilde{I}$ such that $\pi'(i) = \pi(j)$. That is, the assignment of each agent in $\tilde{I}$ in matching π' is assigned to another agent in $\tilde{I}$ in matching π. Suppose not. Then, either $\pi'(i) = \pi(k)$ where $k \in I \backslash \tilde{I}$ or $\pi'(i)$ is unfilled in matching π. The former case cannot be true because of the strict preferences and Pareto domination between π and π'. That is, $\pi'(k) = \pi(k)$ for all $k \in I \setminus \tilde{I}$. The latter case cannot be true because it contradicts the fairness of matching π. Recall that we incorporate non-wastefulness into the definition of fairness.

Then for each $i \in \tilde{I}$ there exists another $j \in \tilde{I}$ such that $\pi'(i) = \pi(j)$. Due to the fairness of matching π, j has higher priority than i for house $\pi(j)$, i.e. $j \succ_{\pi(j)} i$. Since the quota of each house is one, then j is assigned to a different house in matching π'. As a consequence of our claim above, there exists another agent $l \in \tilde{I}$ such that $\pi'(j) = \pi(l)$ and $l \succ_{\pi(l)} j$. Due to finiteness, we eventually have a cycle as in the statement of Theorem 14. Then the trades that make all agents in $\tilde{I}$ should make the housing office worse off. Otherwise, π would fail to be Pareto efficient. That is, the housing office should strictly prefer π to π'.

The condition in Theorem 14 can be explained in words as follows:

Whenever there exists a cycle[18] in the priority structure, the housing office always prefers the student with higher priority to be assigned to that house.

When we use a unique f while determining the priority orders, if we can find a cycle that consists of n students then at least $n-1$ one of them should be existing tenants. In this case, the preferences of the housing office satisfy the requirement given in Theorem 14.

3.5 Simulations

In this section, we compare the performance of DA and TTC in a random environment using computer simulations. In the previous section, we showed that there is no dominance between the two mechanisms in terms of cost efficiency. Using simulations, we will show whether one of these mechanisms outperforms the other one in our setup. We develop our setup by taking some important aspects of the house allocation problem into account. For instance, moving from one house to another may be costly for the existing tenants. Hence, an existing tenant may prefer to stay in her current house even if there is another house that she slightly prefers to her current house. Moreover, some houses may be preferred more than others by all agents. That is, there may be a correlation in agent preferences. We incorporate these points in our definition of the preferences of the agents over the houses.

Let $U_{i,h}$ be the utility of agent $i \in I$ for house $h \in H$. It is defined as:

[18] A cycle is an ordered list of houses and students $(i_n, h_n, i_{n-1}, h_{n-1}, ..., i_1, h_1)$ such that i_k has higher priority than i_{k-1} for house h_k for all $k \in \{1, 2, ..., n\}$ and $i_0 = i_n$.

$$U_{i,h} = \beta \times (\alpha \times Z(h) + (1-\alpha) \times Z(i,h)) + (1-\beta) \times own(i) \times owner(i,h)$$

where $\alpha, \beta \in [0,1]$.[19] The correlation in the agent preferences is captured by α. Parameter β captures the tendency of the existing tenants to keep their current house. $Z(h)$ is an i.i.d standard uniformly distributed random variable and represents the common tastes of agents on house h. $Z(i,h)$ is also an i.i.d standard uniformly distributed random variable and represents the tastes of agent i on house h. Existing agent i's value for keeping his current house is denoted by $own(i)$ and its value is drawn from a standard uniform distribution. The last term in the utility equation, $owner(i,h)$, is equal to 1 if i is currently living in house h and 0 otherwise.

In our simulations, we set the number of students and houses to n and the number of existing tenants to e. Then, given the number of students and the number of existing tenants we index the students by $i = 1, 2, ..., n$ and houses by $h = 1, 2, ...n$. We give the ownership of house k to agent k where $k \leq e$. Then we determine the priority structure from the ownership profile and an ordering that is drawn randomly. We generate random variables $Z(h)$, $Z(i,h)$ and $own(i)$, and by using these random variables and predetermined α and β we determine the utilities. For each setup we run the two mechanisms and calculate the number of existing tenants keeping their current houses in

[19]Erdil and Ergin (2008) define utilities in their simulations similarly.

the outcomes of these mechanisms. By keeping the ownership profile and utilities the same, we run the mechanisms 1,000 times by using different random orderings of agents.

In Figure 3.1, Figure 3.2, and Figure 3.3 we show how many times (out of 1,000) each mechanism beats the other one in terms of cost efficiency for different values of α and β in a house allocation problem consisting of 20 agents (houses). We run our simulations for different numbers of existing tenants. Our simulation results show that the DA mechanism performs much better than the TTC mechanism in terms of cost efficiency. When we look at our simulation results in more detail we have the following observations. When $\alpha = 1$, perfect correlation, there is no dominance between the two mechanisms since they mostly select the same outcome. This is a consequence of perfect correlation in the preferences. As β decreases, the number of runs that either mechanism is performing better decreases. Intuitively, when β is low, existing tenants are ranking their current house at the upper portion of their preference list and they most likely prefer to keep their current house. Another observation is that as the number of existing tenant increases, the number of runs where we observe a domination by either mechanism increases. We also observe that TTC performs better for the lower values of α, i.e. when the correlation is low.

In Figure 3.4, Figure 3.5, and Figure 3.6 we show the average number of existing tenants assigned to their own house by each mechanism for different values of α and β in a house allocation problem consisting of 20 agents (houses).

Our simulation results show that for any values of α and β the average number of existing tenants assigned to their own house under TTC is less than or equal to that under DA. For the lower values of β the difference between the average number of existing tenants assigned to their own house under DA and TTC is small. We observe a larger difference for the higher values of β. Intuitively, as β decreases it is highly probable that existing tenants are ranking their own houses at the top of their list. Another observation is that when $\alpha = 1$ the average number of existing tenants keeping own house is the same in both DA and TTC. This is a consequence of the perfect correlation in the preferences. We also observe that the average number of existing tenants keeping own house increases as α increases under both mechanisms. This is another consequence of the correlation in agent preferences.

3.6 Conclusion

In the one-sided matching problems, mechanisms have been compared based on the welfare of the agents. An example for the one-sided matching problems is the house allocation problem with existing tenants. On-campus housing is one real-life application of this problem. The random serial dictatorship (RSD) mechanism and its variants are used by many colleges in the US. Previous studies, which include only the welfare of the agents, suggest that the housing offices should replace the RSD mechanism with either the deferred acceptance (DA) or the top trading cycles (TTC) mechanism. In contrast with these earlier studies, in this paper we include the welfare of the housing office. We assume that each move of an existing tenant is costly for the housing office and therefore the housing office prefers that existing tenants keep their own houses. We compare the two competing mechanisms based on their costs to the housing office. Although there is no dominance between the two mechanisms, DA has more desirable features in terms of cost efficiency (being less costly) for the housing office. Our simulation results also suggest that the housing office should choose DA instead of TTC.

We also introduce a new class of assignment problem in which the housing office is included in the set of agents. The housing office has preferences over the allocations but since it is defined based on some predetermined rules, i.e. number of moves, the housing office is not a strategic agent. This new class of problem is in between the one-sided and two-sided matching problems. We call it the 1.5-sided matching problem. We show that it inherits some features of both the one-sided and two-sided matching problems. For instance, as in the one-sided problems DA is strategy-proof in the 1.5-sided matching problem. On the other hand, as in the two-sided problems every fair mechanism is Pareto efficient.

In this paper, we focus on the implications of including the welfare of the central authority in the house allocation problem. Our model can be extended where the housing office is strategic and can determine the ordering used to prioritize the agents. Another possible extension can be to analyze the problem in a dynamic environment where an agent can reapply to change his current house more than once.

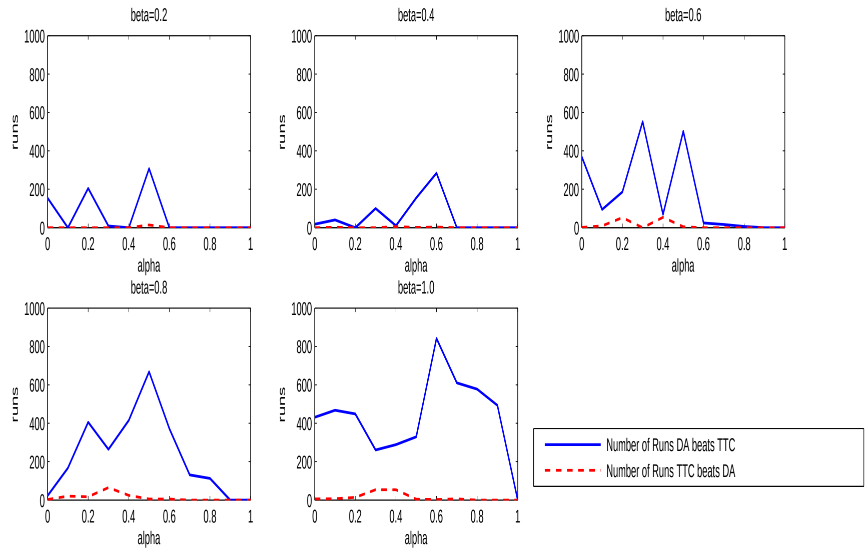

Figure 3.1: Simulations with 20 Houses (Agents) and 5 Existing Tenants

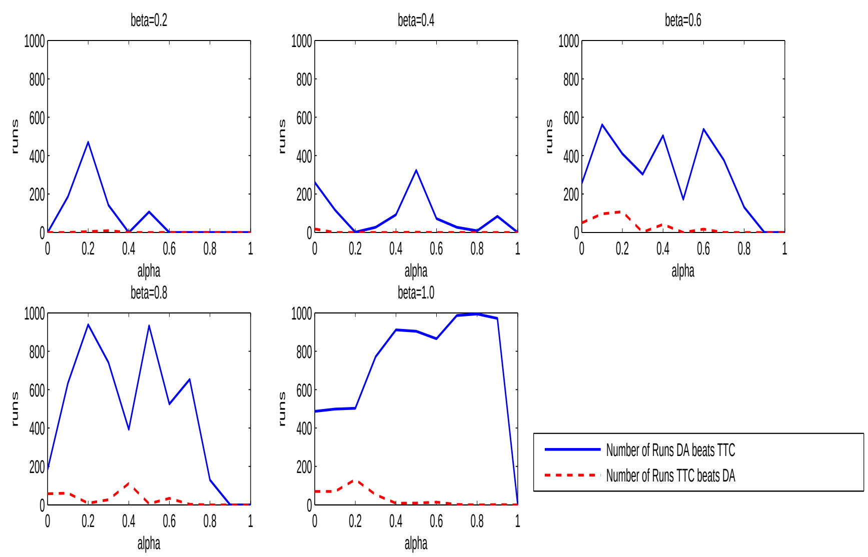

Figure 3.2: Simulations with 20 Houses (Agents) and 10 Existing Tenants

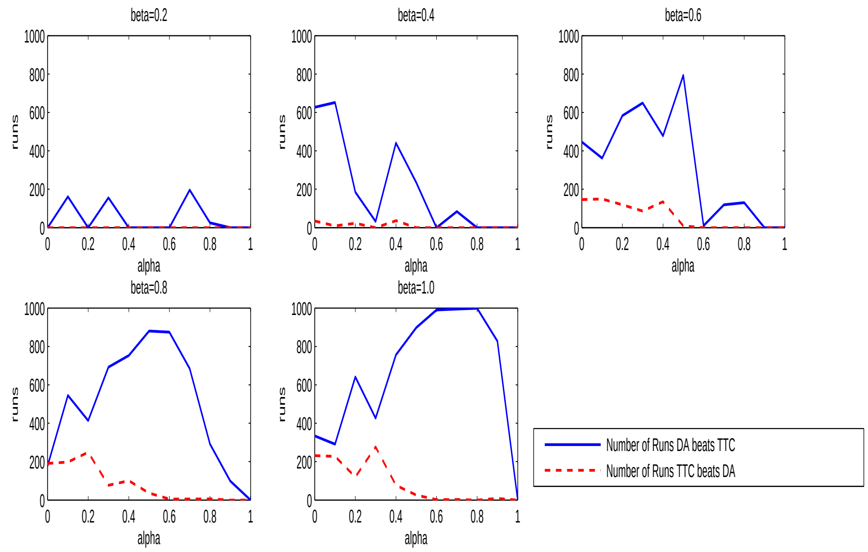

Figure 3.3: Simulations with 20 Houses (Agents) and 15 Existing Tenants

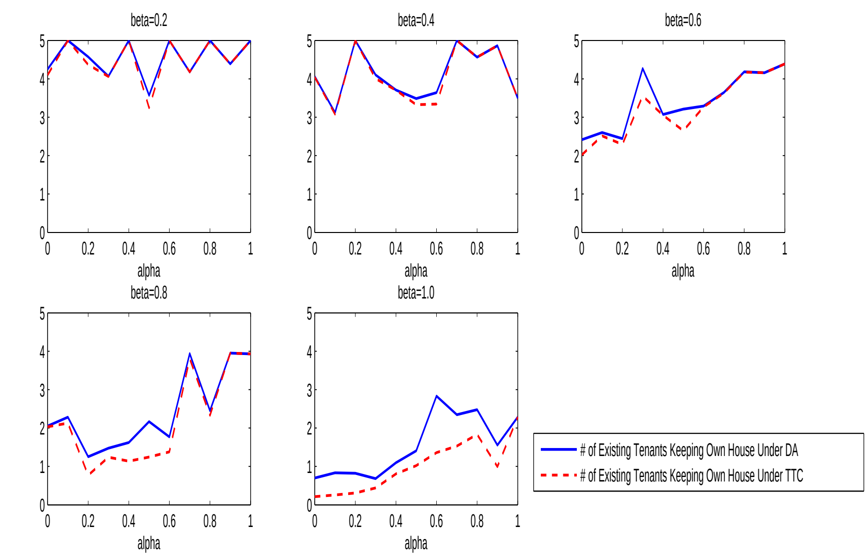

Figure 3.4: Simulations with 20 Houses (Agents) and 5 Existing Tenants

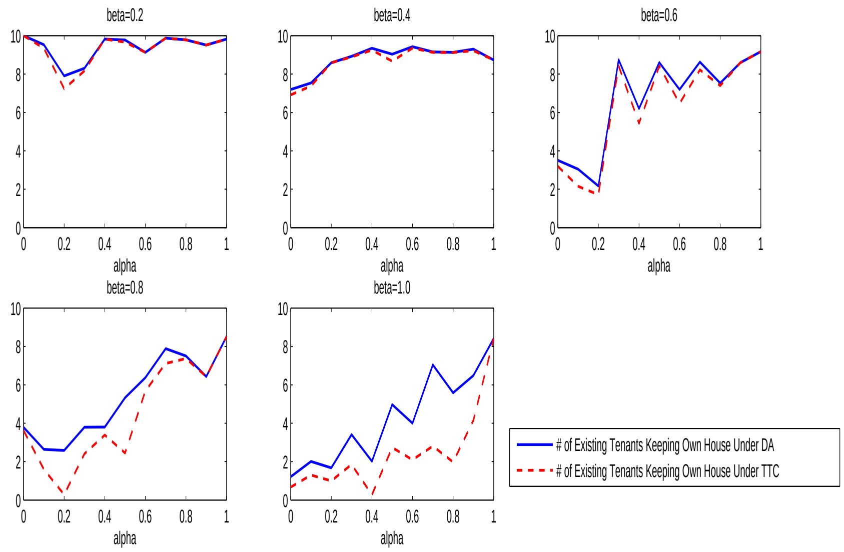

Figure 3.5: Simulations with 20 Houses (Agents) and 10 Existing Tenants

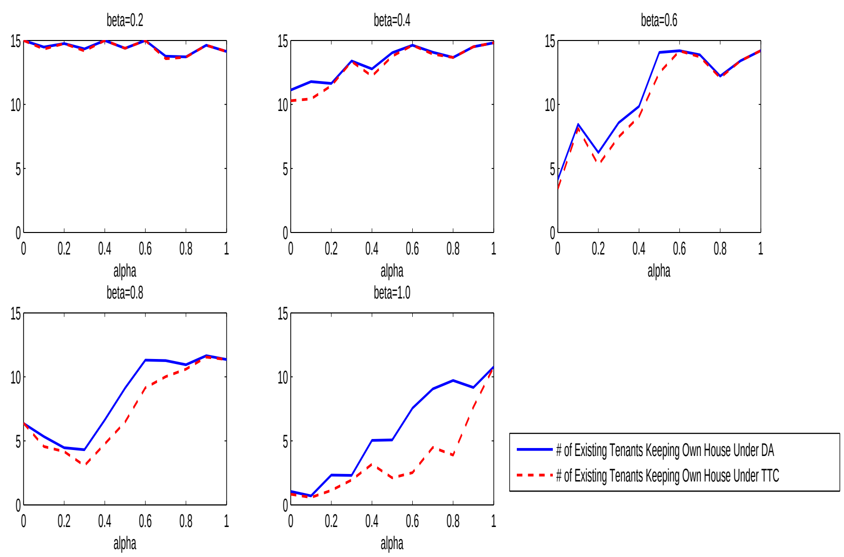

Figure 3.6: Simulations with 20 Houses (Agents) and 15 Existing Tenants

Appendices

Appendix A

Chapter 1 Appendix

A.1 Tuition Exchange Programs and Tuition Co-ops

There are more than eight independent tuition exchange programs helping employees and their families access tuition benefits at member colleges. Each tuition exchange program varies significantly in its structure, benefits, and costs to institutions and students. We have already explained the details of the largest program, The Tuition Exchange, Inc. (TTEI). In this appendix, we provide details of other leading programs, including Council of Independent Colleges Exchange, Catholic College Cooperative Tuition Exchange, Great Lakes Colleges' Association, Associated Colleges of the Midwest, Associated Colleges of the South, Faculty and Staff Children Exchange Program, and Council for Christian Colleges and Universities Tuition Waiver Exchange Program.

The Council of Independent Colleges Tuition Exchange Program (CIC-TEP): CIC-TEP is an extensive exchange program composed of approximately 400 small to mid-sized colleges across the nation, although most colleges are based in the Midwest or the East Cost. All full–time employees of a CIC-TEP participating institution are eligible for the benefit, along with their spouses and dependents. Colleges determine their own policies and guidelines for the eligibility of their own employees. Each participating college is required to accept a minimum of three new students per year. There is no limitation on the number of exported students and all eligible students can apply for the scholarship. Students should be admissible at the importing institution in order to be considered for the scholarship.

After the determination of the sponsored students, the exporting institution completes the tuition exchange participation form and directs the form to the institutions to which their sponsored students are applying. Each student also applies for admission directly to the institutions of her choice and submits all required financial aid information.

Catholic College Cooperative Tuition Exchange (CCCTE): There are 70 members comprised of different schools throughout the country, some large and some small. Each member defines its own benefit levels for tuition exchange. Each institution determines who among its employees is eligible to apply for tuition exchange at another member college. All dependents must be accepted to the college or university of their choice before applying for the benefit. Acceptance at the institution does not guarantee the availability of the benefit. Each participating college is not allowed to import more than 5 students per academic year over the number of students it exports to other participating institutions. There is no limitation on the number of students it exports in a given academic year. Hence, each employee who is determined to be eligible can apply for the scholarship.

Great Lakes Colleges' Association (GLCA): GLCA is a consortium of thirteen private liberal arts colleges in Pennsylvania, Michigan, Ohio, and Indiana. The eligibility of the student's parent is determined by the college at which the parent is employed. All other policies affecting the student are determined by the college the student attends. Under this program, each participant's family will pay a fee equal to 15% of the GLCA mean tuition. The remaining 85% is paid by the exporting institution. Thus there are no concerns about maintaining a balance. Hence, it is no longer an exact exchange program; previously, the association was using an exchange program. After problems related to maintaining balances surfaced, they converted to the current model.

Associated Colleges of the Midwest (ACM): This program is composed of fourteen liberal arts colleges in Wisconsin, Minnesota, Iowa, Illinois, and Colorado. Eligibility is set by the exporting college. Anyone eligible for the importing college's tuition remission program is considered eligible for the program, and she is placed in the applicant pool.

Each participating ACM college compensates 50% tuition to all imported students from participating colleges. Each exporting college determines the level of benefit it offers its employees, though it must be at least 80% of the importing college's tuition. If the exporting college benefit is 80%, the family is responsible for the remaining 20% of tuition. The exporting college might choose a benefit of 90% or 100% or any other level, as long as it is at least 80% of the tuition of the college to be attended.

The current ACM exchange replaced a previous exchange that resulted in imbalances between the member colleges. Since the current system does not have a balancing feature, popular colleges with few qualified employees with dependent students could find themselves with a greater number of incoming students.

Faculty and Staff Children Exchange Program (FACHEX): There are 28 participating Jesuit colleges in this program. The student first must submit a regular application for admission to the FACHEX–participating college involved. This must be done in accordance with that college's regular admission requirements and procedures. Eligibility to participate in the FACHEX Program does not qualify a student for admission, nor does admission qualify a student for FACHEX tuition remission. No institution is obligated to enroll more than three FACHEX students over the number it exports.

Council for Christian Colleges and Universities Tuition-Waiver Exchange Program (CCCU-TWEP): There are 100 participating colleges in this program. Each participating college agrees to accept at least one (with a recommendation of three) new student from other participating institutions. Applicant students apply directly to the institutions of their choice and must meet normal admissions requirements in order to be considered for the exchange program, due to limited space.

A.2 Erasmus Student Exchange Program

The Erasmus program is a student exchange program between universities of the member countries of the European Union.[1] Close to 3 million students have participated since it started in 1987. The number of students benefiting from the program is increasing each year; in 2011, more than 231,408 students attended a college in another member country as an exchange student. The number of member universities is more than 4,000. The Erasmus program aims to improve the quality of higher education and strengthen its European dimension. Considering the growing numbers of exchange students, we can say that the program is fulfilling its purpose.

Students from member universities can participate in the program. Each university from the member countries has the right to apply to be a member institute. All membership applications of universities are sent to the European Commission. Once a university is awarded with membership, it needs to sign bilateral agreements with the other member institutions. In particular, the student exchanges are done between the member universities that have signed a bilateral contract with each other.

The bilateral agreement includes information about the number of students that will be exchanged between the two universities in a given period. Even if the two universities agree to import and export the same number of students, they may fail to fulfill the agreement. The program does not di-

[1] Turkey and Norway are the non-EU members of the program.

rectly punish the universities that run a negative balance. Instead the other universities may prefer not to sign contracts with universities with a negative balance or they may set the number of students to be exchanged very low in their bilateral agreements. Alternatively, the imbalance between two universities can be carried to the new contract that they sign after the termination of the initial one.

The selection process of the exchange students is mostly done as follows. The maximum number of students that can be exported to a partner university is determined based on the bilateral agreement with that partner and the number of students who have been exported since the agreement was signed. The students submit their list of preferences over the partner universities. The university ranks its own students based on predetermined criteria, e.g., GPA and seniority. Based on the ranking, a serial dictatorship mechanism is applied to place students at the available slots. Finally, the list of the students who received the slots of the partner universities is sent to the partners. The partner universities most likely accept all the students on the list.

An exchange student pays her tuition to her own college, not the one importing her. This may lead to financial issues when a college imports more than it exports. For instance, when a college exports 10 students and imports 20 students, it receives tuition for only 10 students and spends money for 20 students. To prevent this financial issue, colleges may be precautionary and set the maximum number of students to be exchanged at a low level.

When we consider the balance of countries we observe that some coun-

	2004/05	2005/06	2006/07	2007/08	2008/09	2009/10	2010/11	Total
UK	9052	9,264	9,273	8,452	8,636	8,770	8,927	62,374
Sweden	3,928	4,532	4,827	5,403	5,793	6,060	6,348	36,891
Denmark	2,087	2,684	2,958	3,292	3,625	3,934	4,262	22,842
Poland	-6,058	-6,911	-7,489	-7,744	-7,256	-6,079	-4,640	-46,177
Germany	-5,154	-5,959	-6,006	-5,752	-5,685	-6,102	-6,059	-40,717
Turkey	-843	-2,024	-3,117	-4,475	-4,560	-5,117	-5,209	-25,345

Table A.1: Balance of Member Countries

tries are running huge positive balances over the years and while others are running huge negative balances (Table A.1). Between 2004 and 2011, the UK imported 62,374 students more than it exported; while Poland exported 46,177 students more than it imported.

This huge imbalance between countries can be easily solved if the 2S-TTC mechanism is adopted. Adoption of the 2S-TTC mechanism requires the market to be centralized. When we look at the numbers of students exchanged between the UK and other countries, we see that the UK always imports more than it exports. Then it is clear that each student from the UK is assigned to her top-choice country.[2] If we assume that the number of students from the UK preferring a college in each member country as their first choice is less than the number of students from that country preferring the UK, then none of the students in the UK is made worse off due to the adoption of the centralized 2S-TTC mechanism.

[2]This is true if we assume that students have preferences over countries.

A.3 Proofs

We start by reminding the reader of the student–proposing deferred–acceptance (DA for short) algorithm of Gale and Shapley (1962). The algorithm works for a given problem $[q, e, P]$ in iterative steps:

Student–Proposing Deferred–Acceptance (DA) Algorithm:

Step 1: Each eligible student makes an offer to her most preferred college. If there is no such a college she is tentatively matched with the null college.

Each college that receives an offer tentatively accepts all best acceptable offers up to its import quota according to its preferences. Any unacceptable offer or any offer not honored due to the import quota constraint is rejected permanently.

In general,

Step k: Each eligible student who does not have a tentatively accepted offer from the previous step makes an offer to the best acceptable college that has not rejected her yet. If there is no such college, she is tentatively matched with the null college.

Each college that holds tentatively accepted offers or receives new offers in this step tentatively accepts all best acceptable offers, among the new and previously held ones, up to its quota according to its preferences. Any unacceptable offer or any offer not honored due to the import quota constraints is rejected permanently.

The algorithm terminates when all eligible students are tentatively accepted either by a college or the null college. Tentatively accepted offers are finalized as matches. Gale and Shapley (1962) (and Roth, 1985) showed that this algorithm finds a stable matching when college preferences over imports are responsive.

We first state and prove the following Lemma, which is used in proving Proposition 7 and Theorem 2:

Lemma 1 *When a college c sets q_c and e_c for its import and eligibility quotas, suppose π is a stable matching of this problem. It later sets $\tilde{e}_c = e_c + 1$ and $\tilde{q}_c$ such that if $|\pi(c)| = q_c$ then $\tilde{q}_c = q_c$, and otherwise $\tilde{q}_c \geq q_c$. Suppose $\tilde{\pi}$ is a stable matching of the second problem. Then we have $b_c^{\tilde{\pi}} \in \{b_c^{\pi} - 1, b_c^{\pi}\}$ and $b_{c'}^{\tilde{\pi}} \in \{b_{c'}^{\pi}, b_{c'}^{\pi} + 1\}$ for all $c' \in C \setminus c$.*

Let π be a stable matching for $[q, e, P]$. Note that $M_c^{\pi} = \pi(c)$, as no student finds her home school acceptable, and π is stable. Denote the newly certified student of college c by i in problem $[(\tilde{q}_c, q_{-c}), (e_c + 1, e_{-c}), P]$.

The number of positions filled by each college is the same at every stable matching by Proposition 6. Thus, without loss of generality, we assume π to be the outcome of the DA mechanism.

First consider the problem $[(\tilde{q}_c, q_{-c}), (e_c, e_{-c}), P]$. If $b_c^{\pi} < 0$, then adding new seats to an under-demanded college will not change the set of students assigned to c in the DA mechanism will select the same outcome for problems $[(q_c, q_{-c}), (e_c, e_{-c}), P]$ and $[(\tilde{q}_c, q_{-c}), (e_c, e_{-c}), P]$. Otherwise, $\tilde{q}_c = q_c$ by assumption.

We will use the sequential DA algorithm introduced by McVitie and Wilson (1971) where the new agent i will be considered at the end, to find the DA outcome for problem $[(\tilde{q}_c, q_{-c}), (e_c + 1, e_{-c}), P]$. Let $\tilde{\pi}$ be the outcome of the DA mechanism for $[(\tilde{q}_c, q_{-c}), (e_c + 1, e_{-c}), P]$.

Let $C_<$ be the set of colleges that could not fill all their seats, and $C_=$ be the set of colleges that did, in π. Formally, $C_< = \{c \in C \ : \ |\pi(c)| < q_c\}$ and $C_= = \{c \in C \ : \ |\pi(c)| = q_c\}$. Observe that if c has a negative balance at μ, then $c \in C_<$; otherwise $c \in C_=$ and $q_c = \tilde{q}_c$. Now, when it is the turn of i to apply in the sequential DA, the current tentative matching is π.

After i starts making offers in the algorithm, let $\tilde{c}$ be the first college that does not reject i. Observe that $\tilde{c} \neq c$ as i does not consider c, her home college, to be acceptable.

In the rest of the proof, as we run the sequential DA, we run the following cases iteratively, starting with student i.

1. If $\tilde{c} = c_\emptyset$, then the algorithm terminates; $b^{\tilde{\pi}} = b^{\pi}$.

2. If $\tilde{c} \in C_<$, then i will be assigned to $\tilde{c}$ and the algorithm terminates; $b^{\tilde{\pi}}_c = b^{\pi}_c - 1$, $b^{\tilde{\pi}}_{\tilde{c}} = b^{\pi}_{\tilde{c}} + 1$, and $b^{\tilde{\pi}}_{-\{c,\tilde{c}\}} = b^{\pi}_{-\{c,\tilde{c}\}}$.

3. If $\tilde{c} \in C_=$, then student $\tilde{i}$ that has the lowest priority among the students in $\pi(\tilde{c})$ is rejected in favor of i. We consider two cases:

   (a) Case $\tilde{i} \in S_c$: The net balance of no college will change since the beginning, and we continue from the beginning above again using student $\tilde{i}$ instead of i.

   (b) Case $\tilde{i} \notin S_c$: The instantaneous balance of c will deteriorate by 1 as i is tentatively accepted. Now, it is $\tilde{i}$'s turn in the sequential DA to make offers. In this series of offers, suppose the first college that does not reject student $\tilde{i}$ is $\tilde{\tilde{c}}$. Denote the home college of $\tilde{i}$ by c' (note that $c' \neq c$).

      i. If $\tilde{\tilde{c}} \in c_\emptyset \cup (C_< \setminus c)$, then the algorithm will terminate, and $b_c^{\tilde{\pi}} = b_c^{\pi} - 1$. Two cases are possible about $\tilde{\tilde{c}}$:

         A. Case $\tilde{\tilde{c}} = c_\emptyset$: We have $b_{c'}^{\tilde{\pi}} = b_{c'}^{\pi} + 1$ and $b^{\tilde{\pi}}_{-\{c,c'\}} = b^{\pi}_{-\{c,c'\}}$.

         B. Case $\tilde{\tilde{c}} \neq c_\emptyset$: We have $b_{\tilde{\tilde{c}}}^{\tilde{\pi}} = b_{\tilde{\tilde{c}}}^{\pi} + 1$ and $b^{\tilde{\pi}}_{-\{c,\tilde{\tilde{c}}\}} = b^{\pi}_{-\{c,\tilde{\tilde{c}}\}}$.

      ii. If $\tilde{\tilde{c}} = c$, then $\tilde{i}$ will be assigned to c and the algorithm will terminate; as the instantaneous change in the balance of c is $+1$, the total change since the beginning will be 0; and no other school's college's balance will change either: $b^{\tilde{\pi}} = b^{\pi}$.

      iii. If $\tilde{\tilde{c}} \in (C_= \setminus c)$, then the lowest priority student held by $\tilde{\tilde{c}}$ will be rejected in favor of $\tilde{i}$. Let this student be $\tilde{\tilde{i}}$. There are two further cases:

A. Case $\tilde{\tilde{i}} \in S_c$: The instantaneous balance of c will increase by 1, and we will start from the beginning again above with $\tilde{\tilde{i}}$ instead of i. The total change in c's balance since the beginning will be 0. Also no other college's balance has changed since the beginning.

B. Case $\tilde{\tilde{i}} \notin S_c$: We start from Step 3(b) above with student $\tilde{\tilde{i}}$ instead of $\tilde{i}$.

Thus, whenever we continue from the beginning, the instantaneous balance of c is b_c^π, and whenever we continue from Step 3(b), the instantaneous balance of c is $b_c^\pi - 1$ and the instantaneous balances of all other colleges are given by vector b_{-c}^π. Due to finiteness, the algorithm will terminate at some point at Steps 1 or 2, or Steps 3(b)i or 3(b)ii; and the net balance of c at the new DA outcome will be b_c^π or $b_c^\pi - 1$. Moreover, whenever the algorithm terminates, the balance of any other college has gone up by one or stayed the same.

We are ready to prove the results stated in the main text:

[Proof of Proposition 7] Let π be the outcome of the DA mechanism in problem $[q, e, P]$ with $q_c \geq e_c$ and $\tilde{\pi}$ be the outcome of the DA mechanism in problem $[(\tilde{q}_c, q_{-c}), (e_c + 1, e_{-c}), P]$ where $\tilde{q}_c \geq q_c$. By Proposition 6, it is sufficient to prove the proposition for π and $\tilde{\pi}$, as all stable matchings have the same balance. Note that $M_c^\pi = \pi(c)$, since no student finds her home school as acceptable, and π is stable.

Two cases are possible:

- $b_c^\pi < 0$: We have $\pi(c) = |M_c^\pi| < |X_c^\pi| \le e_c \le q_c$. Then, by Lemma 1, $b_c^{\tilde{\pi}} \in \{b_c^\pi - 1, b_c^\pi\}$.

- $b_c^\pi \ge 0$: We have two cases again:

  - $|\pi(c)| < q_c$ or $\tilde{q}_c = q_c$: By Lemma 1, $b_c^{\tilde{\pi}} \in \{b_c^\pi - 1, b_c^\pi\}$.

  - $|\pi(c)| = q_c$ and $\tilde{q}_c = q_c + k$ for $k > 0$: Denote the newly certified student of college c by i in problem $[(\tilde{q}_c, q_{-c}), (e_c + 1, e_{-c}), P]$. We first consider the outcome of the DA mechanism in problem $[(\tilde{q}_c, q_{-c}), (e_c, e_{-c}), P]$, which we denote by π''. We first show that the number of students imported by c in matching π'' cannot be less than the one in π. Let $C_<$ be the set of colleges with unfilled capacity in matching π, i.e. $C_< = \{\tilde{c} \in C : |\pi(\tilde{c})| < q_{\tilde{c}}\}$. Due to the non-wastefulness of π, $\pi(s) P_s \tilde{c}$ for all $s \in E \setminus \pi(\tilde{c})$ and $\tilde{c} \in C_<$. We know the DA mechanism is resource monotonic: when the number of seats increases then every student will be weakly better off (cf. Kesten, 2006). That is, $\pi''(s) R_s \pi(s)$ for all $s \in E$. By combining resource monotonicity and individual rationality of the DA mechanism, we can say if a student is assigned to a college in π then he will also be assigned to a college in π''. Hence, we can write:

$$\sum_{c' \in C} |\pi''(c')| \ge \sum_{c' \in C} |\pi(c')|. \tag{A.1}$$

    Note that the difference between the left–hand side and the right–hand side of the equation can be at most one. This follows from the fact that in matching π'' no new student will be assigned to a college in $\tilde{C}$, the number of students assigned to other colleges can increase only for c, and the maximum increment is one.

By combining the non-wastefulness and resource monotonicity we can write:

$$\sum_{\tilde{c}\in C_<} |\pi''(\tilde{c})| \leq \sum_{\tilde{c}\in C_<} |\pi(\tilde{c})|. \tag{A.2}$$

Then, if we subtract the left–hand side of Equation A.2 from the left–hand side of Equation A.1 and the right–hand side of Equation A.2 from the right–hand side of Equation A.1, we get the following inequality:

$$\sum_{c'\in C\setminus C_<} |\pi''(c')| \geq \sum_{c'\in C\setminus C_<} |\pi(c')|. \tag{A.3}$$

Given that each college in $C \setminus C_<$ fills its seats in π, when we subtract $\sum\limits_{c'\in C\setminus(C_<\cup c)} q_{c'}$ from both sides of Equation A.3 we get the following inequality:

$$|\pi''(c)| + \sum_{c'\in C\setminus(C_<\cup c)} (|\pi''(c')| - q_{c'}) \geq |\pi(c)|. \tag{A.4}$$

The term $\sum\limits_{c'\in C\setminus(C_<\cup c)} (|\pi''(c')| - q_{c'})$ is non-negative since $|\pi''(c')| \leq q_{c'}$ for all $c' \in C \setminus (C_< \cup c)$. Therefore, $|\pi''(c)| \geq |\pi(c)|$.

If $|\pi''(c)| = |\pi(c)|$ then $|\pi''(c')| = |\pi(c')|$ for all $c' \in C \cup \{c\}$. This follows from Equation A.4, Equation A.2, and the fact that $|\pi''(c')| \leq |\pi(c')|$ for all $c' \in C \setminus \{c\}$. Therefore, c cannot export and import more students and $b_c^{\pi''} = b_c^{\pi}$. If $|\pi''(c)| > |\pi(c)|$ then at most k more students can be assigned to a college in π'' among the eligible students who were not assigned to a college in π. It is possible that some of students belong to S_c. Thus, $b_c^{\pi''} \in \{b_c^{\pi}, ..., b_c^{\pi} + k\}$.

Thus, by Lemma 1, as we increase the export eligibility quota of college c by 1 and keep the import quota at $\tilde{q}_c$, we have $b_c^{\tilde{\pi}} \in \{b_c^{\pi''} - 1, b_c^{\pi''}\}$, and hence, $b_c^{\tilde{\pi}} \in \{b_c^{\pi} - 1, b_c^{\pi}, ..., b_c^{\pi} + k\}$.

[Proof of Theorem 2] To prove this theorem, we consider two problems: $[(q_c, q_{-c}), (e_c, e_{-c}), P]$ and $[(q'_c, q_{-c}), (e_c - 1, e_{-c}), P]$ with $q_c \geq e_c$ and $q_c \geq q'_c \geq e_c - 1$ such that for college c, $b_c^{\mu} < 0$ for a stable matching μ of the first problem. Let μ' be a stable matching of the second problem. We want to show that $b_{-c}^{\mu} \geq b_{-c}^{\mu'}$ where μ' is an arbitrary stable matching for problem $[(q'_c, q_{-c}), (e_c-1, e_{-c}), P]$. From Proposition 7, we know that $b_c^{\mu'} < 0$ or $b_c^{\mu'} = 0$. By Proposition 6, without loss of generality we assume that μ is the outcome of the sequential DA algorithm for $[(q_c, q_{-c}), (e_c, e_{-c}), P]$ and μ' is the outcome of the sequential DA algorithm for $[(q'_c, q_{-c}), (e_c - 1, e_{-c}), P]$. We have two cases:

Case 1: $b_c^{\mu'} < 0$**.** We have $|\mu'(c)| = |M_c^{\mu'}| < |X_c^{\mu'}| \leq e_c - 1 \leq \min\{q_c, q'_c\}$. Hence, as c did not fill its import quota at μ' under both q_c and q'_c, in problem $[(q_c, q_{-c}), (e_c - 1, e_{-c}), P]$, μ' will still be the outcome of DA. Then when we add a new student i from college c to the set of eligible students, we obtain problem $[(q_c, q_{-c}), (e_c, e_{-c}), P]$. By Lemma 1, we have $b_{c'}^{\mu} \in \{b_{c'}^{\mu'}, b_{c'}^{\mu'} + 1\}$ for all $c' \in C \setminus c$.

Case 2: $b_c^{\mu'} = 0$. There are two possibilities: (1) $|\mu'(c)| < q'_c$ and (2) $|\mu'(c)| = q'_c$.

1. If $|\mu'(c)| < q'_c$, then by Lemma 1, we have $b_{c'}^{\mu} \in \{b_{c'}^{\mu'}, b_{c'}^{\mu'} + 1\}$ for all $c' \in C \setminus c$.

2. If $|\mu'(c)| = q'_c$, then $|\mu'(c)| = e_c - 1 = q'_c$. We first increase the import quota of c from q'_c to q_c and keep its export eligibility quota at $e_c - 1$. Suppose the number of students assigned to college c increases at the outcome of the DA mechanism under $[q_c, (e_c - 1, e_{-c}), P]$, which we denote by μ'', i.e., $|\mu''(c)| > |\mu'(c)| = e_c - 1$. Thus, $b_c^{\mu''} > 0$. When we increase also the export eligibility quota of c from $e_c - 1$ to e_c, then by Lemma 1, $b_c^{\mu} \in \{b_c^{\mu''} - 1, b_c^{\mu''}, b_c^{\mu''} + 1\}$, and hence, $b_c^{\mu} \geq 0$. However, this contradicts the fact that $b_c^{\mu} < 0$. Therefore, $|\mu''(c)| = |\mu'(c)| = q'_c \leq q_c$. Hence, under both problems $[q'_c, (e_c - 1, e_{-c}), P]$ and $[q_c, (e_c - 1, e_{-c}), P]$, DA chooses the same matching, i.e., $\mu'' = \mu'$. When we increase the export eligibility quota of c from $e_c - 1$ to e_c and keep the import quota at q_c, the DA outcome changes from $\mu'' = \mu'$ to μ. By Lemma 1, we have $b_{c'}^{\mu} \in \{b_{c'}^{\mu'}, b_{c'}^{\mu'} + 1\}$ for all $c' \in C \setminus c$.

In either case, $b_{-c}^{\mu'} \leq b_{-c}^{\mu}$.

[Proof of Proposition 9]Consider a graph consisting of students pointing to their assignments under μ, if there is one; and each such student is pointed to by her home college. If a student is assigned to her home college then that college can form only a trivial cycle, and a student in a trivial cycle is neither an import nor an export. Therefore, they do not affect the balance profile. Now consider the other students who are assigned to colleges other than their home colleges. If each assigned student s is in a cycle with another student from $S_{\mu(s)}$ then μ is balanced.

Consider the other direction. Suppose μ is a balanced matching that is not a null matching. First, note that a student who is assigned to her home college forms a trivial cycle. Remove all these students. If we consider the remaining ones, we still have a balanced matching. We set $\tilde{C} = \emptyset$ and start with a college c and a student $s \in S_c$ such that $\mu(s) \in C$. We add c to $\tilde{C}$, i.e., $\tilde{C} = \{c\}$. We select another student s' from $S_{\mu(s)}$ such that $\mu(s') \in C \setminus \mu(s)$. Such a student exists by the balancedness of μ. If $\mu(s') \in \tilde{C}$ then $\mu(s') = c$ and we have a cycle $(c, s, \mu(s), s')$. In this case, we remove the students in the cycle from the problem; the remaining matching is still balanced, and hence, we set $\tilde{C} = \emptyset$, and restart by considering the remaining graph. On the other hand, if $\mu(s') \notin \tilde{C}$ then we update $\tilde{C}$ by including $\mu(s')$, i.e., $\tilde{C} = \{c, \mu(s')\}$. Then, we take a student $s'' \in S_{\mu(s')}$ such that $\mu(s'') \in C \setminus \mu(s')$. Such a student exists by balancedness. If $\mu(s'') \in \tilde{C}$ then we have a cycle including some of the colleges in $\tilde{C} \cup \mu(s'')$ together with some of the students already encountered and s''. We remove the students involved in the cycle from the problem. The remaining matching is still balanced. If $\mu(s'') \notin \tilde{C}$, we update $\tilde{C}$ by including $\mu(s'')$, $\tilde{C} = \{c, \mu(s'), \mu(s'')\}$. Due to finiteness, we will eventually have a cycle. We continue similarly, and as balancedness is maintained whenever we remove students in a cycle, eventually $\tilde{C}$ should be the empty set and all students should be removed from the problem by balancedness.

[Proof of Proposition 3]**Individual Rationality and Respect for Internal Priorities:** We prove individual rationality and respect for internal priorities for 2S-TTC and for the more general 2S-TTCC that we introduce for problems that allow colleges to maintain balance within a predetermined interval. 2S-TTCC also possesses these desirable properties. Individual rationality and respect for internal priorities for 2S-TTC follow from Theorem 16 in Appendix A.4.

Balanced–efficiency: Since the matching selected by 2S-TTC consists of trade cycles in which students and their assignments form unique cycles, its outcome is balanced by Proposition 9. Let π be the matching selected by 2S-TTC. Let $S(\text{k})$ be the set of students assigned in Round k of 2S-TTC. We will prove that π cannot be Pareto dominated by another balanced matching in two parts.

Part I: We first prove that π cannot be Pareto dominated by another individually rational balanced matching. If $s \in S(1)$ then $\pi(s)$ is the highest ranked college in her preference list that considers her acceptable. That is, no agent $s \in S(1)$ can be assigned to a better college considering her acceptable. If there exists a matching ν such that $\nu(s) \succ_s \pi(s)$ then $\nu(s)$ considers i unacceptable. That is, π cannot be Pareto dominated by another ν in which at least one student in $S(1)$ is better off under ν and all students are assigned to a college that considers them acceptable.

If a student $s \in S(2)$ is not assigned to a more preferred college that considers her acceptable, then that college should fill its quota in Round 1 by another student s'. Given s' is assigned in Round 1, $\pi(s')$ should be her favorite college among the ones considering her acceptable. That is, in any matching ν in which s is assigned to $\pi(s')$, student s' will be made worse off. Hence, π cannot be Pareto dominated by another balanced matching ν in which at least one student in $S(2)$ is better off under ν.

We similarly show the same for all other rounds of 2S-TTC. Thus, no student can be assigned to a better college without harming any other student among the colleges that consider her acceptable. Hence, no college can be made better off without harming another college either, if we focus on matchings that are individually rational.

Part II: Next we show that there does not exist an individually irrational balanced matching that Pareto dominates π. To the contrary of the claim, suppose there exists an individually irrational balanced matching ν that Pareto dominates π. Then each $i \in C \cup S$ weakly prefers $\nu(i)$ to $\pi(i)$ and at least one agent $j \in C \cup S$ strictly prefers $\nu(j)$ to $\pi(j)$. Due to the individual rationality of 2S-TTC, every student weakly prefers her assignment in π to being unassigned or to being assigned to an unacceptable college. Therefore, every assigned student in π is also assigned to an acceptable college under ν. Thus, due to the balancedness of both π and ν, $|v(c)| \geq |\pi(c)|$ for all $c \in C$. As ν is individually irrational, there exists some college c_0 such that $s_0 \in \nu(c_0)$

is unacceptable for c_0. As $\nu(c_0) R_{c_0} \pi(c_0)$, there should be at least one student $s_1 \in \nu(c_0) \setminus \pi(c_0)$ such that s_1 is acceptable for c_0 by responsiveness of import preferences and $\nu(s_1) P_{s_1} \pi(s_1)$. We need to consider two cases regarding $\pi(s_1)$:

1. First, suppose $\pi(s_1) = c_\emptyset$. Denote the home college of s_1 by c_1. Hence, $|\nu(c_1)| > |\pi(c_1)|$ by balancedness of ν and π. As $\nu(c_1) \neq \pi(c_1)$, by responsiveness of import preferences and $\nu(c_1) R_{c_1} \pi(c_1)$, there exists a student $s_2 \in \nu(c_1) \setminus \pi(c_1)$ such that s_2 is acceptable for c_1. We also have $\nu(s_2) P_{s_2} \pi(s_2)$.

2. Next, suppose $\pi(s_1) \in C$. Denote $\pi(s_1)$ by c_1. As $\nu(c_1) \geq \pi(c_1)$, there exists $s_2 \in \nu(c_1) \setminus \pi(c_1)$ and s_2 is acceptable for c_2 due to the responsive import references and $\nu(c_1) R_{c_1} \pi(c_1)$. We also have $\nu(s_2) P_{s_2} \pi(s_2)$.

We continue with student s_2 and her assignment $\pi(s_2)$, similarly construct c_2 and then s_3. As we continue, by finiteness, we should encounter the same student $s_k = s_\ell$ for some $k > \ell \geq 1$, that is, we encountered her before in the construction. Consider the students $s_{\ell+1}, s_{\ell+2}, ..., s_k$. Let $s_{k'}$ be the student who is assigned in the earliest round of 2S-TTC in this list. By definition she points to $\pi(s_{k'})$. However, she prefers college $c_{k'-1}$ to her assignment and she is acceptable at $c_{k'-1}$. Moreover, we know that college $c_{k'-1}$ has not been removed yet from the algorithm, as if $c_{k'-1}$ was constructed in Case 1 above then $q_{c_{k'-1}} > |\pi(c_{k'-1})|$ and $s_{k'-1} \in S_{c_{k'-1}}$ is still not removed, and if $c_{k'-1}$ was constructed in Case 2 above then $s_{k'-1} \in \pi(c_{k'-1})$ is still not removed.

Therefore, $s_{k'}$ should have pointed to $c_{k'-1}$ in 2S-TTC in that round. This is a contradiction to ν Pareto dominating π.

[Proof of Theorem 6] Here, we use a variant of the 2S-TTC mechanism. In this variant, we only select one cycle in one round. If there is more than one cycle, then the selection is done randomly. Let $S(\mathrm{k})$ be the set of students assigned in Round k of this variant of 2S-TTC. Suppose the theorem does not hold.

Let ψ be the mechanism satisfying all four axioms and select a different matching for problem $[q, e, P]$. Denote the outcome of 2S-TTC for problem $[q, e, P]$ by μ. Since both 2S-TTC and ψ are balanced–efficient then there exist at least two students such that one of them prefers her assignment in $\psi[q, e, P]$ to the one in μ and the other student prefers her assignment in μ to the one in $\psi[q, e, P]$.

We first prove the following claim:

Claim: If there exists a student in $S(\mathrm{k})$ who prefers her assignment in $\psi[q, e, P]$ to the one in μ, then there exists another student in $\bigcup_{\mathrm{k}'=1}^{\mathrm{k}\text{-}1} S(\mathrm{k})$ who prefers her assignment in μ to the one in $\psi[q, e, P]$.[3]

We use induction in our proof. Consider the students assigned in Round 1 of 2S-TTC. If $S(1)$ is a singleton then the student in $S(1)$ is assigned to $c_{\emptyset}$.

[3]We take $\bigcup_{\mathrm{k}'=1}^{0} S(\mathrm{k}') = \emptyset$.

Any college that she prefers to $c_\emptyset$ considers her unacceptable. If she prefers her assignment under ψ to $c_\emptyset$ then she is assigned to a college considering her unacceptable by ψ. Therefore ψ is individually irrational. If she prefers $c_\emptyset$ to her assignment under ψ then ψ is individually irrational. Then any individually rational mechanism will assign her to $c_\emptyset$. If $S(1)$ is not a singleton then all students in $S(1)$ are assigned to the best colleges considering themselves as acceptable and they prefer their assignment in μ to $c_\emptyset$. If a student prefers her assignment in $\psi[q, e, P]$ to her assignment in μ then ψ is individually irrational. Hence, all students in $S(1)$ weakly prefer their assignment in μ.

In the inductive step, assume that for all Rounds 1,...,k-1, for some k>1, the claim is correct. Consider Round k. If there exists a student $s \in S(\mathrm{k})$ who prefers college $c = \psi[q, e, P](s)$ to $\mu(s)$ then c considers s acceptable and its seats are filled in Rounds 1,...,k-1 of 2S-TTC or s is unacceptable for c. In the latter case, ψ is individually irrational. Consider the former case. Let student s' be assigned to c under μ in a Round $\mathrm{k}' \leq$ k-1 but not under $\psi[q, e, P]$, as s is assigned instead of her. If she prefers her assignment in c to $\psi[q, e, P](s')$ then we are done. If she does not, $\mathrm{k}' >1$, and by the inductive step, there exists a student $s'' \in S(\mathrm{k}'')$ for some $\mathrm{k}'' < \mathrm{k}'$ who prefers $\mu(s'')$ to $\psi[q, e, P](s'')$.

Now we are ready to prove the theorem. Let student $s \in S(\mathrm{k})$ prefer $\mu(s)$ to $\psi[q, e, P](s)$ and $\mu(s') = \psi[q, e, P](s')$ for all $s' \in \bigcup_{\mathrm{k}'=1}^{\mathrm{k}-1} S(\mathrm{k}')$. Such a student exists by the Claim, as $\mu \neq \psi[q, e, P]$.

We will construct our proof in three steps. Assign to each round of

2S-TTC mechanism a counter and set it as $\text{Counter}(\text{k}') = |S(\text{k}')| - 1$ for all k'.

Step 1: Construct a preference profile $\tilde{P}$ as follows: Let student $s \in S_c$ rank only $\mu(s)$ as acceptable in $\tilde{P}_s$ and $\tilde{P}_{-s} = P_{-s}$. 2S-TTC will select μ for problem $[q, e, \tilde{P}]$. Since ψ is student strategy-proof, $\psi[q, e, \tilde{P}](s) = c_\emptyset$.

Then, we check whether the assignments of students in $\bigcup_{\text{k}'=1}^{\text{k-1}} S(\text{k}')$ are the same in $\psi[q, e, \tilde{P}]$ and μ. If not, then for some $\tilde{\text{k}} < \text{k}$, there exists a student $\tilde{s} \in S(\tilde{\text{k}})$ preferring $\mu(\tilde{s})$ to $\psi[q, e, \tilde{P}](\tilde{s})$ and each student in $\bigcup_{\text{k}'=1}^{\tilde{\text{k}}\text{-1}} S(\text{k}')$ gets the same college in μ and $\psi[q, e, \tilde{P}]$. Then repeat this step by taking $P := \tilde{P}$, $s := \tilde{s}$, and $\text{k} := \tilde{\text{k}}$.

This repetition will end by the finiteness of rounds and the fact that $\bigcup_{\text{k}'=1}^{0} S(\text{k}') = \emptyset$. When all students in $\bigcup_{\text{k}'=1}^{\text{k-1}} S(\text{k}')$ get the same college in μ and $\psi[q, e, \tilde{P}]$ then we proceed to Step 2.

Step 2: In Step 1, we have shown that s prefers $\mu(s)$ to $\psi[q, e, \tilde{P}](s)$. Suppose c is the home college of s. Set $\tilde{e}_c$ equal to the rank of student s in her home college's internal priority order, that is, $\tilde{e}_c = r_c(s)$, and let $\tilde{e}_{-c} = e_{-c}$. In the problem $[q, \tilde{e}, \tilde{P}]$, 2S-TTC assigns all students in $\bigcup_{\text{k}'=1}^{\text{k}} S(\text{k}')$ to the same college as in μ. Student s is unassigned in $\psi[q, \tilde{e}, \tilde{P}]$ since ψ respects internal priorities. We check whether the assignments of students in $\bigcup_{\text{k}'=1}^{\text{k-1}} S(\text{k}')$ are the same in both $\psi[q, \tilde{e}, \tilde{P}]$ and μ. If not, then by the Claim, there should exist a student $\tilde{s} \in S(\tilde{\text{k}})$ preferring $\mu(\tilde{s})$ to $\psi[q, \tilde{e}, \tilde{P}](\tilde{s})$ and each student in $\bigcup_{\text{k}'=1}^{\tilde{\text{k}}\text{-1}} S(\text{k}')$ gets the same college in μ and $\psi[q, \tilde{e}, \tilde{P}]$ where $\tilde{\text{k}} < \text{k}$; then we restart from

Step 1 by taking $P := \tilde{P}$, $s := \tilde{s}$, $\mathrm{k} := \tilde{\mathrm{k}}$, and $e := \tilde{e}$.

Eventually, by the finiteness of the rounds of 2S-TTC and as we reduce the round k in each iteration, we reach the point in our proof such that students in $\bigcup_{\mathrm{k}'=1}^{\mathrm{k}\text{-}1} S(\mathrm{k}')$ get the same college in μ and $\psi[q, \tilde{e}, \tilde{P}]$.

Observe that s is the last remaining eligible student of the home college c in Round k of 2S-TTC for problem $[q, \tilde{e}, \tilde{P}]$ by the choice of $\tilde{e}_c$ being equal to the ranking of s in $\succ_c$. As for all $s' \in \bigcup_{\mathrm{k}'=1}^{\mathrm{k}\text{-}1} S(\mathrm{k}')$ we have $\mu(s') = \psi[q, \tilde{e}, \tilde{P}](s')$ and $\psi[q, \tilde{e}, \tilde{P}](s) = c_\emptyset$, the student $s' \in S(\mathrm{k}) \cap \mu(c)$ will be assigned to a different college in $\psi[q, \tilde{e}, \tilde{P}]$ than c. Otherwise, ψ fails to be balanced. Moreover, as for all $s'' \in \bigcup_{\mathrm{k}'=1}^{\mathrm{k}\text{-}1} S(\mathrm{k}')$, $\mu(s'') = \psi[q, \tilde{e}, \tilde{P}](s'')$ and s' points to the best available college that finds her acceptable in Round k of 2S-TTC, $c = \mu(s')\tilde{P}_{s'}\psi[q, \tilde{e}, \tilde{P}](s')$. We decrease Counter(k) by 1. If Counter(k) > 0 then we turn back to Step 1 by taking $P := \tilde{P}$ and $s := s'$; and otherwise continue with Step 3.

Step 3: By construction above, each student $\tilde{s} \in S(\mathrm{k})$ ranks only $\mu(\tilde{s})$ as acceptable in $\tilde{P}_s$ and she is the last certified student by her home college in problem $[q, \tilde{e}, \tilde{P}]$. In Step 2, we showed that there exist at least 2 students $s_1, s_2 \in S(\mathrm{k})$ who are not assigned to $\mu(s_1)$ and $\mu(s_2) = c_1$, respectively, in $\psi[q, \tilde{e}, \tilde{P}]$, where c_1 is the home college of s_1. Then, they are assigned to $\psi[q, \tilde{e}, \tilde{P}]$, by the individual rationality of ψ. Recall that in 2S-TTC for $[q, \tilde{e}, \tilde{P}]$, each student certified by the home colleges of s_1 and s_2 – colleges c_1 and c_2, respectively – other than s_1 and s_2 are removed in a round earlier than k. Suppose for student $s_3 \in S(\mathrm{k})$, $\mu(s_3) = c_2$. Since $\psi[q, \tilde{e}, \tilde{P}](s_2) = c_\emptyset$, for all

$\tilde{s} \in \bigcup_{\mathrm{k}'=1}^{\mathrm{k}\text{-}1} S(\mathrm{k}')$, $\psi[q, \tilde{e}, \tilde{P}](\tilde{s}) = \mu(\tilde{s})$ (by Step 2), and ψ is balanced, student s_3 cannot be assigned to c_2 in $\psi[q, \tilde{e}, \tilde{P}]$, and hence, $\psi[q, \tilde{e}, \tilde{P}](s_3) = c_\emptyset$. We continue similarly with s_3 and home college of s_3, say college c_3, eventually showing that for all $\tilde{s} \in S(\mathrm{k})$, $\psi[q, \tilde{e}, \tilde{P}](\tilde{s}) = c_\emptyset$. Recall that students in $S(\mathrm{k})$ had formed a trading cycle in which each agent in the cycle was assigned the home college of the next student in the cycle in μ. Thus, $\psi[q, \tilde{e}, \tilde{P}]$ is Pareto dominated by the balanced matching ν obtained as $\nu(\tilde{s}) = \psi[q, \tilde{e}, \tilde{P}](\tilde{s})$ for all $\tilde{s} \in S \setminus S(\mathrm{k})$ and $\nu(\tilde{s}) = \mu(\tilde{s})$ for all $\tilde{s} \in S(\mathrm{k})$; that is, ν is obtained from $\psi[q, \tilde{e}, \tilde{P}]$ by students in $S(\mathrm{k})$ trading their assignments with each other to get their assignments in μ. This contradicts the balanced–efficiency of ψ. Hence, $\psi[q, e, P] = \mu$, i.e., ψ is equivalent to 2S-TTC.

The following Lemma is used in proving Theorem 5 and Theorem 10:

Lemma 2 *Let π and $\tilde{\pi}$ be the matching selected by 2S-TTC for problems $[q, e, P]$ and $[(\tilde{q}_c, q_{-c}), (\tilde{e}_c, e_{-c}), P]$ where $\tilde{q}_c \leq q_c$ and $\tilde{e}_c \leq e_c$. $M_c^{\tilde{\pi}} \subseteqq M_c^{\pi}$ and $X_c^{\tilde{\pi}} \subseteqq X_c^{\pi}$.*

We have two cases to consider:

Case 1: $\tilde{q}_c \leq q_c$ and $\tilde{e}_c < e_c$. We consider the case in which one more student is certified by college c. Denote the student added to the eligible set by s. Let $r_c(s') = r_c(s) - 1$. Consider the execution of the 2S-TTC mechanism for this new problem. If college c imports $\tilde{q}_c$ students before student s's turn then college c will be removed and certifying one more student will not affect the set of students exported and imported by c. Now consider the case in which college c imports less than $\tilde{q}_c$ before student s's turn. Denote the intermediate matching that we have just after s' is assigned by ν. Since c is removed just after the student s' is assigned in problem $[(\tilde{q}_c, q_{-c}), (\tilde{e}_c, e_{-c}), P]$, $M_c^{\tilde{\pi}} = M_c^{\nu}$ and $X_c^{\tilde{\pi}} = X_c^{\nu}$. If s is assigned to a college $c' \in C \setminus c$, college c will import one more acceptable student. Denote that matching by μ. Given one more student is exported and imported compared to the ones in ν we have $M_c^{\pi} = M_c^{\nu} \subsetneqq M_c^{\mu}$ and $X_c^{\pi} = X_c^{\nu} \subsetneqq X_c^{\mu}$. If s is assigned to the null college, then college c will have the same import and export set. If we keep certifying students until e_c we will have $M_c^{\pi} \subseteqq M_c^{\mu'}$ and $X_c^{\pi} \subseteqq X_c^{\mu'}$ where μ' is the matching selected by 2S-TTC for problem $[(\tilde{q}_c, q_{-c}), e, P]$. Note that it is also possible to have $M_c^{\pi} \subsetneqq M_c^{\mu'}$ and $X_c^{\pi} \subsetneqq X_c^{\mu'}$ if one of the student with internal rank between $\tilde{e}_c$ and e_c is assigned to a college in C in matching μ'.

Case 2: $\tilde{q}_c < q_c$ and $\tilde{e}_c = e_c$. Let π and $\tilde{\pi}$ be the matchings that the 2S-TTC mechanism selects for problems $[q, e, P]$ and $[(\tilde{q}_c, q_{-c}), e, P]$, respectively. If $|M_c^{\nu}| < \tilde{q}_c$ then 2S-TTC will select the exactly same cycles when college c reports its true import quota. Therefore, $M_c^{\pi} = M_c^{\tilde{\pi}}$ and $X_c^{\pi} = X_c^{\tilde{\pi}}$. If $|M_c^{\nu}| = \tilde{q}_c$ and if c leaves the market after all its eligible students are considered, then it will not make a difference if college c reports its true quota. If $|M_c^{\nu}| = \tilde{q}_c$ and if c leaves the market before all its eligible students are considered, then at least one more student $s \in S_c$ will be considered in 2S-TTC. As in the previous case college c may import and export at least one more student. At the end we get $M_c^{\tilde{\pi}} \subsetneqq M_c^{\pi}$ and $X_c^{\tilde{\pi}} \subsetneqq X_c^{\pi}$ if some of the students who are considered only when c reports q_c are assigned. Otherwise, $M_c^{\pi} = M_c^{\tilde{\pi}}$ and $X_c^{\pi} = X_c^{\tilde{\pi}}$.

[Proof of Theorem 5]Take a problem $[q, e, P]$. Take a college c. Suppose that preference reports are fixed such that c does not report any of unacceptable students as acceptable in these reports. We have two cases to consider for possible quota manipulations by college c:

Case 1: College c reports $\tilde{q}_c \leq q_c$ and $\tilde{e}_c \leq |S_c|$: In Lemma 2 we have shown that when college c reports its import and certification quota as higher, the set of students imported by college c (weakly) expands. Given that college preferences are responsive over the incoming students, then reporting $\tilde{q}_c \leq q_c$ and $\tilde{e}_c \leq |S_c|$ is weakly dominated by reporting true import quota and certifying all students.

Case 2: College c reports $\tilde{q}_c > q_c$: This strategy is weakly dominated by reporting its true import quota q_c. We show this as follows: Let ν and μ be the matchings that the 2S-TTC mechanism selects when c reports $\tilde{q}_c$ and q_c, respectively. If $|M_c^{\nu}| \leq q_c$, then $M_c^{\nu} = M_c^{\mu}$ and $X_c^{\nu} = X_c^{\mu}$; thus, it is indifferent between the two matchings. However, if $|M_c^{\nu}| > q_c$, as import preferences are only responsive up to true import quota and μ is individually rational, then it prefers μ to ν.

Lemma 3 and 4 are used in proving Proposition 13:

Lemma 3 *Let μ_k be the matching selected in some Round k of RDA and $\tilde{e}_c$ be the number of students certified by c when RDA terminates in problem*

$[q, e, P]$. If $b_c^{\mu_k} > 0$, then $\tilde{e}_c \geq \min\{e_c^k + b_c^{\mu_k}, e_c\}$ where e_c^k is the number of students certified by college c in Round k.

Suppose not. Since $\tilde{e}_c < \min\{e_c^k + b_c^{\mu_k}, e_c\}$, the number of additional students certified from college c until the end of the RDA after Round k is less than $b_c^{\mu_k} > 0$. Thus, by Proposition 12, the balance of c cannot fall more than $b_c^{\mu_k}$ after Round k until the end of RDA, and hence c runs a positive balance at the end of RDA. Since $\tilde{e}_c < \min\{e_c^k + b_c^{\mu_k}, e_c\}$, college c has additional students to certify when RDA terminates, which contradicts the termination condition of RDA.

Lemma 4 *Let μ_k be the matching selected in some Round k of RDA and e_c^k be the number of students certified by college c in that round. Let $\nu_{k,m}$ be the tentative matching selected in some Step m of the sequential DA algorithm in Round k of RDA. Let $e_c^{k,m}$ be the number of students sponsored by college c who already made offers in the sequential DA and got matched to either a real college or null college in $\nu_{k,m}$. If $b_c^{\nu_{k,m}} - e_c^k + e_c^{k,m} > 0$ then $b_c^{\mu_k} \geq b_c^{\nu_{k,m}} - e_c^k + e_c^{k,m} > 0$.*

In each step of the sequential DA an unassigned student is considered and a student will be removed from her seat only if another one replaced her. That is, in the sequential DA the number of students assigned to college c in Step m+1 cannot be less than what it is in Step m, i.e., $|\nu_{k,m}(c)| \leq |\nu_{k,m+1}(c)|$. Therefore, $|M_c^{\nu_{k,m}}| \leq |M_c^{\mu_k}|$. In Round k of RDA, the maximum number of students exported from college c can be $|X_c^{\nu_{k,m}}| + (e_c^k - e_c^{k,m})$. Then $b_c^{\mu_k} \geq |M_c^{\nu_{k,m}}| - |X_c^{\nu_{k,m}}| - (e_c^k - e_c^{k,m}) = b_c^{\nu_{k,m}} - e_c^k + e_c^{k,m}$.

[Proof of Proposition 13]Consider two different orders (selection rules) in RDA. Denote the outcome of the first order by μ' and the outcome of the second one by μ''. The outcome of RDA equals the outcome of the DA mechanism for the given set of certified students in the last round. Denote the set of eligible students in the last round of the two orders by E' and E'', respectively. Let k be the first round that different students are certified. Given that the sets of students certified in the previous rounds, $E(\mathrm{k}-1)$, are the same, we have $\mu'_{k-1} = \mu''_{k-1}$. Let s' and s'' be the students certified in round k by two different selection rules, respectively. By Lemma 3, we have $\{s', s''\} \subset E' \cap E''$. Moreover, any students certified in Round k are in $E' \cap E''$. Then consider the last round of the first selection rule. We run the sequential DA in the last round. The outcome of the sequential DA is independent of the order of students considered (McVitie and Wilson, 1971). We consider students one by one, and if a student who was tentatively accepted in an earlier step is rejected, we consider her in the next step; otherwise, we consider a student who has not been considered yet. We first consider students in $E(\mathrm{k}-1)$. Then we consider student s''. In some later steps we will get the matching μ''_k. By Lemmas 3 and 4 the set of students who are guaranteed to be certified as a consequence of μ''_k should be in E'. Therefore, the student who is certified in Round k based on the second selection rule is in E'. Then we consider her in the next step of the sequential DA. We continue similarly and show that any student certified based on the second selection rule is in E'. That is $E'' \subseteq E'$. We can also start with considering s' and show that any student certified based on the second selection rule is in E''. Therefore $E' = E''$. Thus, $\mu' = \mu''$.

We first state and prove the following lemma, which is used in proving Theorem 7.

Lemma 5 *If a student $s' \in S_{\tilde{c}}$ is assigned to college c' by RDA in problem $[q, e, P]$, then she will be assigned to c' by RDA in $[q, e, (\tilde{P}_{s'}, P_{-s'})]$ where $\tilde{P}_{s'}$: $c' - c_{\emptyset}$.*

Let E and $\tilde{E}$ be the set of students who are considered in the last round of RDA for problems $[q, e, P]$ and $[q, e, (\tilde{P}_{s'}, P_{-s'})]$, respectively. First, we show that $\tilde{E} \subseteq E$. First note that certification of student s' does not depend on the preference list she submits. Therefore she is certified in problem $[q, e, P]$ if and only if she is certified in problem $[q, e, (\tilde{P}_{s'}, P_{-s'})]$. We look at the case in which she is certified, $i \in E$ and $i \in \tilde{E}$.

Consider the following variation of the RDA mechanism. In each round, use the sequential DA mechanism and consider s' as the last student if he is an eligible student in that round. Before the turn of student s', calculate the balance of each college. By Propositions 4 and 13, we certify a student from a college $c'' \in C \setminus \{\tilde{c}\}$ if college c'' has a positive balance. Otherwise, if $\tilde{c}$ has a positive balance and s' is not eligible then certify a student from college $\tilde{c}$. If s' is eligible and we cannot certify a student from colleges in $c'' \in C \setminus \{\tilde{c}\}$, then we certify a student from $\tilde{c}$ as long as its balance is greater than 1. Due to finiteness we cannot update the set of eligible students and we need to consider student s'. Denote the set of students certified so far by E_0. Note that $s' \in E_0$. Denote the round that we consider s' by k.

When s' reports $P_{s'}$, due to population monotonicity and the fact that $E_0 \subseteq E$, in Round k s' is assigned to college c'', which is weakly better than c'. Then consider the last round of the problem when s' submits $P_{s'}$. The outcome of RDA when s' submits is equal to the outcome of DA mechanism for problem $[C, E, q, \succ, P]$. Due to strategy-proofness the DA mechanism will assign student s' to c' in problem $[C, E, q, \succ, (\tilde{P}_{s'}, P_{-s'})]$. Due to resource monotonicity s' will be assigned to c' by the DA mechanism in problem $[C, E_0, q, \succ, (\tilde{P}_{s'}, P_{-s'})]$. Note that $[C, E_0, q, \succ, (\tilde{P}_{s'}, P_{-s'})]$ is the problem that we consider in Round k when s' submits $\tilde{P}_{s'}$. Therefore, s' will be assigned to c' in Round k of the RDA mechanism when she submits $\tilde{P}_{s'}$. If the mechanism terminates in this round when s reports $\tilde{P}_s$, then we are done: $\tilde{E} = E_0$ and $\tilde{E} \subseteq E$.

If the mechanism does not terminate in this round when s' reports $\tilde{P}_{s'}$, then we need to check two cases. In the rest of our analysis we consider the sequential DA mechanism and s' is the last student taken into account in Round k. Either s causes rejection of another student, s_1, from college c' or s is assigned to c' and the balance of c' becomes positive.

Case 1: We first show that student s_1 will be rejected by college c' in the last round of RDA for problem $[C, S, q, e, \succ, P]$. Let ν be the tentative matching selected by the sequential DA just before the turn of student s'

in $[C, E_0, q, \succ, (\tilde{P}_{s'}, P_{-s'})]$. When we apply the sequential DA to the problem $[C, S, q, e, \succ, P]$ by first considering the students in $E_0 \setminus s'$ and then the student s', we get the same tentative matching ν just before the turn of s'. Since s_1 is rejected from c' when s' applies in $[C, E_0, q, \succ, (\tilde{P}_{s'}, P_{-s'})]$, it should be the case that $|\nu(c')| = q_{c'}$ and s_1 has the lowest priority among the students in $\nu(c')$. Given the fact that s' is assigned to c' in $[C, S, q, e, \succ, P]$, $s' \notin \nu(c')$ and $|\nu(c')| = q_{c'}$ at least one of the students in $\nu(c')$ has to be rejected. Since s_1 has the lowest priority, she cannot be assigned to s' in the outcome selected for problem $[C, S, q, e, \succ, P]$. Student s_1 will apply to her next choice in both problem $[C, E, q, \succ, P]$ and $[C, E_0, q, \succ, (\tilde{P}_{s'}, P_{-s})]$. We consider the following subcases.

- If her next choice is $c_\emptyset$, then she will be assigned to $c_\emptyset$ and we get the final allocation of the sequential DA mechanism for problem $[C, E_0, q, \succ, (\tilde{P}_{s'}, P_{-s})]$. Moreover, this is the outcome of the Round k of RDA when s' submits $\tilde{P}_{s'}$. Due to the updating procedure described above in matching ν, all the colleges except $\tilde{c}$ either have a non-positive balance or they have reached their certification quota and $b^{\nu}_{\tilde{c}} = 1$. If $s_1 \in S_{\tilde{c}}$, then $\tilde{c}$ has positive balance of 1 in the final allocation selected in Round k when s' submits $\tilde{P}_{s'}$. If $s_1 \in S_{c1 \neq \tilde{c}}$, then $b^{\nu}_{c_1} = 0$ and the balance of c_1 becomes 1. Then one more student will be certified by the home college of s_1. Let s_2 be that student. Suppose $s_2 \notin E$. We apply the sequential DA in problem $[C, E, q, \succ, P]$. We consider students in the same order we described above. When we get matching ν the balance of the home college of s_2 is either 0 or 1. The latter case is true if $s_2 \in S_{\tilde{c}}$. By the definition of the sequential DA the number of students assigned to the home college of s_2 in ν is weakly less than the one in the final allocation. If $s_2 \notin S_{\tilde{c}}$ then the number of students exported by the home college of s_2 will be less than that in ν. This follows from the fact that $\nu(s_1) \in C$ and s_1 is assigned to $c_\emptyset$ in the final allocation. Therefore, the home college of s_2 has a positive balance in the final allocation and has one more student to certify. If $s_2 \in S_{\tilde{c}}$, then the number of students exported by the home college of s_2 cannot be more than that in ν. Given that $b^{\nu}_{\tilde{c}} = 1$, college $\tilde{c}$ has a positive balance in the final allocation and has one more student to certify. Both cases contradict with the termination condition of RDA. Repeat case 1 by taking $s_1 = s_2$.

- If the next choice of s_1 is $c'' \in C$, $|\nu(c'')| = q_{c''}$ and all students in $\nu(c'')$ have higher priority for c'' than s_1 then s_1 will be rejected from c'' when she applies in $[C, E_0, q, \succ, (\tilde{P}_s, P_{-s})]$. She will also be rejected from c'' in $[C, E, q, \succ, P]$. To show this, consider the steps of the sequential DA applied in the last round when s' reports $P_{s'}$. Consider the students in the same order described above. We have shown that s_1 will be rejected from c' and will apply to c'' above. From the definition of the sequential DA all students assigned to c'' when s_1 applies will have at least high priority as the student with the lowest priority in $\nu(c'')$. Given that the student with the lowest priority in $\nu(c'')$ has higher priority than s_1, she cannot be assigned to c'' . Therefore, in both problems s_1 will apply to her next best choice. Repeat case 1 for the next best choice of s_1.

- If the next choice of s_1 is $c'' \in C$ and either $|\nu(c'')| < q_{c''}$ or at least one student in $\nu(c'')$ has lower priority for c'' than s_1 then s_1 will be tentatively accepted when she applies to c'' in $[C, E_0, q, \succ, (\tilde{P}_s, P_{-s})]$. If $|\nu(c'')| < q_{c''}$, then the sequential DA mechanism will terminate and college c'' will have a positive balance and certify one more student in problem $[C, E_0, q, \succ, (\tilde{P}_s, P_{-s})]$. For the same reasons explained in the subcase 1 that student is in E. If there exists a student in $\nu(c'')$ with lower priority than s_1 and $|\nu(c'')| = q_{c''}$, then that student will be replaced with s_1. Denote that student by s_2. She will be also be rejected from the same college in problem $[C, E, q, \succ, P]$. To see this, apply the sequential DA mechanism by first considering students in $E_0 \setminus s'$ in problem $[C, E, q, \succ, P]$. All the other students assigned to c'' are preferred to s_2 by college c''. Then she will be the first agent to be rejected. Since s_1 will apply to c'' in the following steps in problem $[C, E, q, \succ, P]$. Repeat case 1 by taking $s_1 = s_2$.

Case 2: In the second case when s' applies to c', the sequential DA mechanism will terminate and we will get the allocation of Round k of RDA when s' submits $\tilde{P}_{s'}$. As explained above, $b^{\nu}_{c'} = 0$, and it becomes positive when s' applies. Therefore it certifies one more student. We denote that student by $\tilde{s}$. We can show that $\tilde{s}$ is also in E by following the similar steps in the first subcase of case 1.

If we continue, we will see that the student certified when we consider $[C, E_0, q, \succ, (\tilde{P}_s, P_{-s})]$ will be in E. For the next round, update E_0 by adding the new student certified and repeat the same rounds. Therefore, $\tilde{E} \subseteq E$.

In problem $[C, E, q, \succ, (\tilde{P}_s, P_{-s})]$ student s will be assigned to c' by the DA mechanism. Otherwise, strategy-proofness of the DA mechanism would be violated. Moreover, DA will assign s to c' in problem $[C, \tilde{E}, q, \succ, (\tilde{P}_s, P_{-s})]$. Otherwise, resource monotonicity would be violated.

[Proof of Theorem 7]Suppose not. Suppose that $s \in S_c$ can benefit from misreporting her preferences. Let P_s be the true preference of s. Let c be the best college that student s can be assigned by misreporting her preferences. As a consequence of the claim, she will be assigned to c if she submits $\tilde{P}_s : c - c_\emptyset$. We first show that student s will be assigned college c when she submits $\tilde{\tilde{P}}_s : c' - c - c_\emptyset$ where c' is the college just ranked above c in P_s.

We use the variant of RDA defined in Lemma 5. In addition, in step k we take the outcome of step k-1 as an intermediate (tentative) matching of the sequential DA mechanism and continue with the student certified in step k. Let O be the ordered list of students who applied after the application of student i to college c by reporting $\tilde{P}_s$. That is, o_1 is the student who is just considered after the application of i to college c. Student o_1 can be the student who was rejected from college c or the one who is certified because college c has a positive balance after the application of s. Since student s stays in college c, O does not contain student s. Note that a student can be listed in O more than once. Denote the tentative matching that the sequential DA selects just before it is the turn of s by ν. Let $\tilde{O}$ be the ordered list of students who applied after the application of student i to college c' by reporting $\tilde{\tilde{P}}_s$ and before the application of i to c.

Now consider the possible cases when s submits $\tilde{\tilde{P}}_s$.

Case 1: Let all students tentatively assigned to college c', $\nu(c')$, be more preferred by college c'. Then, s will be rejected from c' without changing anything and apply to c. Therefore, she will get c when she adds c'.

Case 2: Let there exist at least one student in $\nu(c')$ who is less preferred to s by college c' or $|\nu(c')| < q_{c'}$. The student rejected from c' or the one newly certified will be $\tilde{o}_1$ and we can find all the students in $\tilde{O}$. Note that s will be in $\tilde{O}$ because s will be rejected from college c'.

Case 2.1: If $\tilde{O}$ and O are disjoint, then when s applies to c she starts the same O list and she will be assigned to c at the end.

Case 2.2: If $\tilde{O}$ and O have common students, we claim that $\tilde{O}$ is a subset of O. After s is rejected from c' then she will apply to c and the rejection list O will start. At some point we reach a student in $\tilde{O}$. Given that $\tilde{O}$ is a "cycle" and all the students have been moved along the cycle, we skip the cycle. We will continue with the same student who is placed just after the cycle $\tilde{O}$ in O. Therefore, nothing will change.

Therefore, s can get c by submitting $\tilde{\tilde{P}}_s : c' - c - c_\emptyset$. We can continue by adding the college just ranked above c' in P_s. We need to check whether the cycle that we will have intersects with $\tilde{O}$ or not. If they intersect, exactly the same students will be in both cycles. Then we repeat the same cases discussed above. Then we will do this for all colleges ranked above c and get the upper portion of the preference list.

[Proof of Proposition 14]RDA terminates when the matching selected in the last round is balanced, or for each college c with a positive balance, $\tilde{e}_c = e_c$. If μ is balanced then $B^\mu = 0$. Since aggregate balance of any other matching cannot be negative, $B^\mu \leq B^\nu$ holds. Thus, suppose μ is not balanced and the second case is true. Apply DA to the problem $[q, e', P]$ (recall that $e' \geq \tilde{e}$). Denote its outcome by ν'.

Next we show that the number of students imported to any college in matching ν' cannot be less than the number of students imported at μ. Suppose this is not true and there exists a college $c' \in C$ where $|M^\mu_{c'}| > |M^{\nu'}_{c'}|$. Recall that, as no student finds her home school acceptable, students imported and assigned are the same for each school under any individually rational matching. Then, there exists a student i such that $\mu(i) = c'$ and $\nu'(i) \neq c'$. Since $q_c \geq |M^\mu_{c'}|$ and $|M^\mu_{c'}| > |M^{\nu'}_{c'}|$ then $q_c > |M^{\nu'}_{c'}|$. Hence by stability of ν' for $[q, e', P]$, student i should prefer $\nu'(i)$ to $c' = \mu(i)$. This contradicts the population monotonicity of DA, as μ, the outcome of RDA, is, by definition, the outcome of DA in problem $[q, \tilde{e}, P]$ while ν' is the outcome of DA for $[q, e', P]$.

Let C_+ be the set of colleges having a positive balance in μ. Recall that for all $c \in C_+$, $\tilde{e}_c = e_c$, and hence, $\tilde{e}_c = e'_c = e_c$. Since DA is population monotonic, a student who is assigned to $c_\emptyset$ in μ is also assigned to $c_\emptyset$ in ν'. Therefore, $X_c^\mu \supseteq X_c^{\nu'}$ for all $c \in C_+$.

For any matching μ', by definition $\sum_{c \in C} b_c^{\mu'} = 0$. Since $\sum_{c \in C_+} |b_c^\mu| = \sum_{c \in C_+} b_c^\mu$ and $\sum_{c \in C \setminus C_+} |b_c^\mu| = -\sum_{c \in C \setminus C_+} b_c^\mu$ (recall that $C \setminus C_+$ is the set of zero and negative balance colleges at μ),

$$\sum_{c \in C \setminus C_+} |b_c^\mu| = \sum_{c \in C_+} |b_c^\mu|. \tag{A.5}$$

On the other hand, $X_c^\mu \supseteq X_c^{\nu'}$ and $|M_c^\mu| \leq |M_c^{\nu'}|$ for all $c \in C_+$ together imply colleges in C_+ have also positive balances at ν' and

$$\sum_{c \in C_+} |b_c^\mu| = \sum_{c \in C_+} b_c^\mu = \sum_{c \in C_+} |M_c^\mu| - |X_c^\mu| \leq \sum_{c \in C_+} |M_c^{\nu'}| - |X_c^{\nu'}| = \sum_{c \in C_+} b_c^{\nu'} = \sum_{c \in C_+} |b_c^{\nu'}|. \tag{A.6}$$

This together with $\sum_{c \in C} b_c^{\nu'} = 0$ imply that

$$\sum_{c \in C \setminus C_+} |b_c^{\nu'}| \geq -\sum_{c \in C \setminus C_+} b_c^{\nu'} = \sum_{c \in C_+} b_c^{\nu'} = \sum_{c \in C_+} |b_c^{\nu'}|. \tag{A.7}$$

Thus, Equations A.5, A.6, and A.7 together with Proposition 6 imply that $B^\mu = \sum_{c \in C} |b_c^\mu| \leq \sum_{c \in C} |b_c^{\nu'}| = B^{\nu'} = B^\nu$.

[Proof of Theorem 8] Suppose there exists a mechanism, φ, satisfying all three properties. Consider the following example. There are five colleges $C = \{a, b, c, d, e\}$ and six export candidates: $S_a = \{\mathbf{1}, \mathbf{2}\}$, $S_b = \{\mathbf{3}, \mathbf{4}\}$, $S_c = \{\mathbf{5}\}$, $S_d = \{\mathbf{6}, \mathbf{7}\}$, and $S_e = \{\mathbf{8}\}$. The true import and export eligibility quotas for each college are: $q_a = 2$, $q_x = 1$ for all $x \in C \setminus a$ and $e_x = |S_x|$ for all

$x \in C$. The internal priority of colleges : $\mathbf{1} \succ_a \mathbf{2}$, $\mathbf{3} \succ_b \mathbf{4}$, $\mathbf{6} \succ_b \mathbf{7}$. All students are acceptable to each college. The preference profile of students is:

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
b	d	a	c	b	c	a	a
e	$c_\emptyset$	$c_\emptyset$	$c_\emptyset$	d	$c_\emptyset$	$c_\emptyset$	$c_\emptyset$
$c_\emptyset$				$c_\emptyset$			

Denote the outcome of mechanism φ by μ. We consider the following cases.

Case 1. $\mu(\mathbf{1}) = b$:

Step 1: We first show that $\mu(\mathbf{3}) = a$.

Suppose not. Then $\mu(\mathbf{3}) = c_\emptyset$. Otherwise individual rationality would be violated. Since $\mu(\mathbf{3}) = c_\emptyset$, the best possible case for college b is that $\mu(\mathbf{4}) = c$. College b prefers any matching in which $\mathbf{3}$ is assigned to a college to a matching in which $\mathbf{3}$ is assigned to $c_\emptyset$, due to responsiveness on outgoing student preferences. Thus, by strategy-proofness for college b, $\mathbf{3}$ would be assigned to $c_\emptyset$ again if college b sets $e_b = 1$. Let this matching be ν. Due to balancedness for b, $\nu(\mathbf{1}) \neq b$, as its only sponsored student $\mathbf{3}$ is assigned to $c_\emptyset$.

By strategy-proofness and individual rationality for $\mathbf{1}$, when $e_b = 1$, if $\mathbf{1}$ reports $P_{\mathbf{1}} = b - c_\emptyset$ (i.e., e as unacceptable), then φ will assign him to $c_\emptyset$.

Now $e_b = 1$, $P_{\mathbf{1}} = b - c_\emptyset$. By strategy-proofness for a in quota manipulation (i.e., for the same reasons mentioned for college b above), if a sets $e_a = 1$, $\mathbf{1}$ should be assigned to $c_\emptyset$. Now $e_a = 1, e_b = 1$, $P_{\mathbf{1}} = b - c_\emptyset$. In this problem $\mathbf{3}$ cannot be assigned to a due to the balancedness for college a, whose only sponsored student $\mathbf{1}$ is assigned to $c_\emptyset$. However, the matching selected in the new problem is Pareto dominated by the matching in which **1** and **3** are assigned to b and a, respectively, and all the other students keep their assignment. A contradiction to balanced-efficiency of φ. Hence we showed that $\mu(\mathbf{3}) = a$.

Step 2: Due to the balancedness for b, which fills its unique quota with **1**, we also have $\mu(\mathbf{4}) = c_\emptyset$.

Step 3: We also claim that $\mu(\mathbf{5}) = d$ and $\mu(\mathbf{6}) = c$.

Note that due to the balancedness for d, which has a unique quota to fill in the original problem, and individual rationality, $\mu(\mathbf{5}) = d$ if and only if $\mu(\mathbf{6}) = c$. If this claim is not true, then both students are unassigned. However, if d sets $e_d = 1$, by strategy-proofness (i.e., for the same reasons we have explained for college b above), then both **5** and **6**, the unique sponsored students of c and d, respectively, remain assigned to $c_\emptyset$. This contradicts balanced efficiency, as **5** and **6** could have been matched to d and c, respectively.

Step 4: All other students are assigned to $c_\emptyset$.

We have shown that in the initial problem, any mechanism satisfying all properties mentioned in the theorem and assigning **1** selects the following matching: Given that $\mu(\mathbf{1}) = b$, we have $\mu(\mathbf{2}) = c_\emptyset$, $\mu(\mathbf{3}) = a$, $\mu(\mathbf{4}) = c_\emptyset$, $\mu(\mathbf{5}) = d$, $\mu(\mathbf{6}) = c$, $\mu(\mathbf{7}) = c_\emptyset$, and $\mu(\mathbf{8}) = c_\emptyset$.

Step 5: Now consider the initial problem. Suppose a reports **3** as unacceptable. Denote the new matching selected in this case by μ'. Due to the individual rationality $\mu'(\mathbf{3}) = c_\emptyset$. We claim that $\mu'(\mathbf{4}) = c$ and $\mu'(\mathbf{5}) = b$.

Due to balancedness and individual rationality, if $\mu'(\mathbf{5}) = b$ then $\mu'(\mathbf{4}) = c$. We consider the following two cases: (1) $\mu'(\mathbf{4}) = c_{\emptyset}$. In this case $\mu'(\mathbf{5}) \neq b$ due to balancedness. If $\mathbf{5}$ announces $P_{\mathbf{5}} = b - \emptyset$, then she will be assigned to $c_{\emptyset}$ due to strategy-proofness. In that case, $\mathbf{4}$ cannot be assigned to any college due to balancedness for c and individual rationality. Therefore, the Pareto–improving trade between $\mathbf{4}$ and $\mathbf{5}$ is omitted. (2) $\mu'(\mathbf{4}) = c$ and $\mu'(\mathbf{5}) \neq b$. Then $\mu'(\mathbf{5}) = d$ by balancedness for c. For similar reasons, if $\mathbf{5}$ announces $P_{\mathbf{5}} = b - \emptyset$, the mechanism violates balanced–efficiency: again, a contradiction. We showed that $\mu'(\mathbf{4}) = c$ and $\mu'(\mathbf{5}) = b$.

Due to balanced–efficiency we have: $\mu'(\mathbf{1}) = e$, $\mu'(\mathbf{2}) = d$, $\mu'(\mathbf{3}) = c_{\emptyset}$, $\mu'(\mathbf{4}) = c$, $\mu'(\mathbf{5}) = b$, $\mu'(\mathbf{6}) = c_{\emptyset}$, $\mu'(\mathbf{7}) = a$ and $\mu'(\mathbf{8}) = a$. College a benefits from misreporting $\mathbf{3}$ as unacceptable, because $X_a^{\mu'} = \{\mathbf{1}, \mathbf{2}\}$ and $X_a^{\mu} = \{\mathbf{1}\}$, and as its preferences are responsive over export students, it prefers μ' to μ.

Case 2. $\mu(\mathbf{1}) = e$: Suppose $\mathbf{1}$ reports $P_{\mathbf{1}} = b - \emptyset$. She will be assigned to $c_{\emptyset}$ due to strategy-proofness. Note that a prefers any matching in which $\mathbf{1}$ is assigned to a college to a matching in which she is assigned to $c_{\emptyset}$, by responsiveness of its preferences over export students. Thus, if a sets $e_a = 1$ (while $P_{\mathbf{1}} = b - \emptyset$) then $\mathbf{1}$ should not be assigned to b and should remain assigned to $c_{\emptyset}$ in order φ not to violate immunity to quota manipulation. Then, in this new problem $\mathbf{3}$ cannot be assigned to a due to balancedness, as $\mathbf{1}$ is the only sponsored student of a and she is assigned to $c_{\emptyset}$. In this new problem, if b sets $e_b = 1$ (that is while $e_a = 1$ and $P_{\mathbf{1}} = b - \emptyset$) then $\mathbf{3}$ should be assigned to $c_{\emptyset}$ due to the immunity to quota manipulation (this is true, as b prefers every matching in which $\mathbf{3}$ is assigned to a college to any matching in which she is assigned to $c_{\emptyset}$, by responsiveness). Moreover, $\mathbf{1}$ should be assigned to $c_{\emptyset}$ due to the balancedness of b and individual rationality for $\mathbf{1}$.

Therefore, a Pareto–improving trade between **1** and **3** being assigned to b and a, respectively, exists. This contradicts the balanced–efficiency of φ.

Case 3. $\mu(\mathbf{1}) = c_{\emptyset}$: Then $\mu(\mathbf{8}) = c_{\emptyset}$ by balancedness for e, as no student is assigned to e in μ by individual rationality. This violates balanced–efficiency. Welfare can be improved by assigning **1** to e and **8** to a.

Case 4. $\mu(\mathbf{1}) \in \{c, d\}$: This violates individual rationality.

[Proof of Proposition 9]We need to show that the 2S-TTC mechanism satisfies all axioms mentioned in the theorem. In Theorem 3 we have shown that the 2S-TTC mechanism is individually rational, balanced–efficient, and respects internal priorities under responsive preferences over the incoming students. The proofs of individual rationality and respecting internal priorities do not depend on any assumption about the preferences of colleges. Similarly, the proof of strategy-proofness for students does not depend on any assumption about the preferences of colleges. Therefore, individual rationality, respecting internal priorities, and strategy-proofness for students of 2S-TTC follow from the proof of Theorem 3 and Theorem 4. In the proof of balanced–efficiency of 2S-TTC under responsive preferences over the incoming students (Theorem 3), we use the responsive preferences in the second part, where we have shown that the outcome of 2S-TTC mechanism cannot be Pareto dominated by a balanced and individually irrational matching. Under the no unacceptable import students assumption, we do not need to use responsiveness over the incoming students, because a college would be always worse off compared to the outcome of the 2S-TTC mechanism when an unacceptable student is assigned to it. Therefore, there does not exist a balanced and individually irrational matching that Pareto dominates the outcome of the 2S-TTC mechanism under the assumption of no unacceptable import students. In order to show that there does not exist a balanced and individually rational matching that Pareto dominates the outcome of 2S-TTC mechanism, we can follow Step 1 of the same proof.

In showing that the 2S-TTC mechanism is the unique mechanism satisfying all these axioms (Theorem 6), we did not use the assumption of responsive preferences over incoming students. Therefore, we can follow the same steps in showing that under the assumption of no unacceptable import students, the 2S-TTC mechanism is the unique mechanism satisfying the axioms.

[Proof of Theorem 10]Take a problem $[q, e, P]$. Take a college c. We have two cases to consider for possible quota manipulations by college c:

Case 1: College c reports $\tilde{q}_c \leq q_c$ and $\tilde{e}_c \leq |S_c|$: In Lemma 2, we have shown that when college c over-reports its import and certification quotas the set of students exported by college c (weakly) expands. Given that college preferences are responsive over the outgoing students, then reporting $\tilde{q}_c$ and $\tilde{e}_c$ is weakly dominated by reporting true import quota and certifying all students.

Case 2: College c reports $\tilde{q}_c > q_c$: This strategy is weakly dominated by reporting its true import quota q_c. We show this as follows: Let ν and μ be the matchings that the 2S-TTC mechanism selects when c reports $\tilde{q}_c$ and q_c, respectively. If $|M_c^\nu| \leq q_c$, then $M_c^\nu = M_c^\mu$ and $X_c^\nu = X_c^\mu$; thus, it is indifferent between the two matchings. However, if $|M_c^\nu| > q_c$, then the feasibility of matching ν is violated.

A.4 Tuition Exchange Problem with Tolerance: Two–Sided Top–Trading–Cycles–and–Chains Mechanism

In the previous sections, we focused on the case where each college is required to have a zero balance. In this appendix, we relax the zero balance constraint and allow colleges to maintain a balance within an interval $[l, u]$ where $l \leq 0 \leq u$.[4] When either l or u equals zero, the problem turns into the case we studied in Section 1.6.

When we allow the colleges to hold a non-zero balance, then there may exist some colleges exporting (importing) more students than they import (export). In this case we cannot represent all the allocations by cycles. Therefore we need to consider chains in addition to the cycles.

Formally, a **chain** is defined as an ordered list of college-student pairs $(c_1, s_1, c_2, s_2, ..., c_k)$ such that:

- college c_1 points to student s_1,
- student s_1 points to college c_2,
- $\vdots$ $\qquad$ $\vdots$

[4] Here, l and u are integers.

- college c_{k-1} points to student s_{k-1},
- student s_{k-1} points to college c_k.

We refer to the college c_1 as the tail and college c_k as the head of the chain. We only consider chain that does not cause the violation of the interval limits, which we call **valid chain**.

In this section we use a mechanism similar to the top–trading–cycles–and–chains (TTCC) mechanism introduced by Roth, Sönmez, and Ünver (2004). We refer to it as the two-sided top-trading-cycles-and-chains mechanism (2S-TTCC). For a given tuition exchange problem and an interval the 2S-TTCC mechanism selects the outcome as follows.

Two-Sided Top Trading Cycles and Chains:

Step 0: Let $\pi(c)$ be the random number assigned to college $c \in C$. Assign two counters for each college $c \in C$, o_c^q and o_c^e, and set them equal to q_c and e_c, respectively. These track remaining quotas for imports and exports, respectively. Let b_c track the current balance of the college c in the fixed portion of the matching. Initially set $b_c = 0$ for each college c. Assign only an export counter for the null college $c_\emptyset$ and set it equal to $|S|$.

Step 1: Each student points to her favorite college in $C \cup c_\emptyset$, which considers her acceptable, and each college $c \in C$ points to the student $s \in S_c$ who has the highest internal priority. The null college $c_\emptyset$ points to the students pointing to it.

Proceed to Step 2 if there is no cycle. Otherwise locate each cycle, and assign each student to the college that she points to.

- The eligible student counter, o_c^e, of each college c whose student is in a cycle is reduced by one.
  - If $o_c^e = 0$ and either $o_c^q = 0$ or $b_c = u$ then remove college c.
  - If $o_c^e = 0$, $o_c^q > 0$ and $b_c < u$ then college c becomes a **passive**[5] college.
- The import counter, o_c^q, of each college c in a cycle is reduced by one only if the cycle includes at least two colleges.
  - If $o_c^q = 0$ and either $o_c^e = 0$ or $b_c = l$ then remove college c.
  - If $o_c^q = 0$, $o_c^e > 0$ and $b_c > l$ then college c is removed from all other students' preferences.
- Return to Step 1.

Step 2: If there are no students left, we are done. If not, then all chains are ending with passive colleges. If $b_c = l$ for each **active**[6] college c, then remove all passive colleges and go to step 1. Otherwise, find the chain whose tail has the lowest π among the active colleges with $b_c > l$. Here π is used as a fixed

[5]A college is passive if it has available quota to import but all its sponsored students are removed. Therefore a passive college cannot point to a student.

[6]A college is active if all of its sponsored students have not yet been removed.

tie-breaker among colleges. Assign each student in that chain to the college that she points to. Denote the tail and head colleges of the chain by c_t and c_h, respectively. Other colleges in the chain are represented by $\tilde{c}$.

- The eligible student and import counters of each college $\tilde{c}$ are reduced by one.
  - If $o^q_{\tilde{c}} = 0$ and either $b_{\tilde{c}} = l$ or $o^e_{\tilde{c}} = 0$, then remove college c.
  - If $o^e_{\tilde{c}} = 0$ and $o^q_{\tilde{c}} > 0$, then college c becomes a passive college.
  - If $o^q_{\tilde{c}} = 0$, $b_{\tilde{c}} > l$ and $o^e_{\tilde{c}} > 0$, then college c is removed from all other students' preferences.
- The eligible student counter and b_{c_t} of college c_t is reduced by one.
  - If $o^q_{c_t} = 0$ and either $b_{c_t} = l$ or $o^e_{c_t} = 0$, then remove college c_t.
  - If $o^e_{c_t} = 0$ and $o^q_{c_t} > 0$, then college c_t becomes a passive college.
- The $o^q_{c_h}$ is reduced by one and b_{c_h} is increased by one.
  - If $o^q_{c_h} = 0$ or $\tilde{b}_c = u$, then remove college c.
- Return to Step 1.

The algorithm terminates when there are no remaining eligible students left. Denote the matching selected by 2S-TTCC by μ.

The 2S-TTCC mechanism inherits the desired features of the 2S-TTC mechanism. In Theorem 15 we show that students cannot benefit from misreporting their preferences and in Theorem 17 we show that its outcome cannot be Pareto dominated by another individually rational matching ν such that $b_c^{\nu} \in [l, u]$.

Theorem 15 *2S-TTCC is strategy-proof for students.*

We use a variation of 2S-TTCC in which only the student with the highest priority points to a college in each round. Let μ be the matching selected by 2S-TTCC under truth-telling. Let $\mathrm{k} > 0$ be the first round that we cannot locate a cycle. Student s assigned in Round $\mathrm{k}' < \mathrm{k}$ (under truth-telling) cannot affect the assignments done in earlier rounds. If s forms a cycle by misreporting in Round $\mathrm{k}'' < \mathrm{k}'$ then she should have pointed to a worse college than $\mu(s)$. Before Round k', all the colleges that s prefers to $\mu(s)$ should have been removed since each student points to the most preferred college among the remaining ones. Therefore, student s cannot get a better college by misreporting.

Now consider Round k. First assume that we have a proper chain. As we mentioned above, any active student in Round k cannot affect the assignments done in earlier rounds. Then consider the student pointed to by the tail college of chain. This student will be assigned in this round no matter which college she points to. Therefore she will point to the most preferred college among the remaining ones. The next students in the chain will do so as well. The other active students in this round cannot affect the assignment of students in the chain without hurting themselves.

Now consider the case where we don't have a proper chain. That is, all the active colleges have a balance of l. Then we will remove all the passive colleges and 2S-TTCC reduces to the 2S-TTC mechanism. It is easy to see that we will not have chains in the future rounds too.

Let $\mathrm{k}'' > \mathrm{k}$ be the first round after Round k that we cannot locate a cycle. Then we can apply the same reasoning that we used for rounds before k and show that all agents assigned in Round $\tilde{\mathrm{k}}$ such that $\mathrm{k} < \tilde{\mathrm{k}} < \mathrm{k}''$ cannot be better off by misreporting. This is also true for Round k''.

In the 2S-TTCC mechanism, a student starts pointing to the colleges in her preference list after all the other students with higher internal rank are assigned to a college including the null college. Moreover, a student points to the colleges ranked over $c_\emptyset$ that consider her acceptable. As a consequence of these two features, the 2S-TTCC mechanism satisfies individual rationality and respects internal ranking.

Theorem 16 *2S-TTCC is individually rational and respects internal priorities.*

Individual Rationality: Students will be assigned to the null college, $c_\emptyset$, when they point to it, and hence, they will never need to point to an unacceptable college. Moreover, a student cannot point to a college that considers her unacceptable. Therefore, none of the unacceptable students for college $c \in C$ will be assigned to a college c. Thus, 2S-TTCC is individually rational.

Respect for Internal Priorities: Suppose, contrary to the claim, that 2S-TTCC does not respect internal priorities. Then there exists a student $i \in S_c$ who is assigned to a college in problem $[(q_c, q_{-c}), (e_c, e_{-c}), P]$, but not assigned to a college in problem $[(\tilde{q}_c, q_{-c}), (\tilde{e}_c, e_{-c}), P]$ where $\tilde{q}_c \geq q_c$ and $\tilde{e}_c > e_c$. Let S(k) and $\tilde{S}$(k) be the set of students assigned in the Round k of 2S-TTCC applied to the problems $[(q_c, q_{-c}), (e_c, e_{-c}), P]$ and $[(\tilde{q}_c, q_{-c}), (\tilde{e}_c, e_{-c}), P]$, respectively. In both problems, the same set of agents will be active in the first step. Since we consider the same preference profile, $S(1) = \tilde{S}(1)$. Then, if $i \in S(1)$, we are done. If not, then consider the second step. Since the same set of students is removed with their assignments then the set of active students and the remaining colleges in the second step of 2S-TTCC applied to the problems will be the same. Moreover, students will be pointing to the same colleges in both problems. Hence, $S(2) = \tilde{S}(2)$. Then, if $i \in S(2)$, we are done. If not, we can repeat the same steps and show that i will be assigned in the matching selected by the 2S-TTCC mechanism in problem $[(\tilde{q}_c, q_{-c}), (\tilde{e}_c, e_{-c}), P]$. (Moreover, she will be assigned to the same college.)

Theorem 17 *For any problem $[q, e, P]$ and tolerance interval $[l, u]$, there does not exist an individually rational matching ν that Pareto dominates the outcome of 2S-TTCC and $l \leq b_c^{\nu} \leq u$ for all $c \in C$.*

Denote the outcome of 2S-TTCC mechanism by μ. Let $\mathrm{k} > 0$ be the first round that we cannot locate a cycle. Without loss of generality, assume $\mathrm{k} > 1$. Again consider the variant that we described in the proof of Theorem 15. In the first round, each student is pointing to her most preferred college. If a student is assigned in this round, then she should get the same college under ν. Now consider students assigned in Round $1 < \mathrm{k}' < \mathrm{k}$. All the colleges that a student preferred to her assignment should have been removed in an earlier round, and we cannot make that student better off without hurting another student assigned in an earlier round. Note that all colleges removed in Round $\mathrm{k}' < \mathrm{k}$ fulfills either certification or the import quota.

Now consider the students assigned in Round k. First, consider the case where there exists a valid chain. All of them are assigned to a college that they prefer most among the remaining ones. They cannot be made better off without making some students assigned in the earlier rounds worse off. If there does not exist a valid chain, then 2S-TTCC reduces to the 2S-TTC mechanism. If college c is removed because of either $b_c^\mu = u$ or $b_c^\mu = l$, then we cannot assign more students to c in ν.

Let $\mathrm{k}'' > \mathrm{k}$ be the first round after Round k that we cannot locate a cycle. Then we can apply the same reasoning that we have used for rounds before k and show that all agents assigned in Round $\tilde{\mathrm{k}}$ such that $\mathrm{k} < \tilde{\mathrm{k}} < \mathrm{k}''$ cannot be better off by misreporting. This is also true for Round k''.

Every student assigned to a college should be assigned the same college in ν. If we assign more students in ν then either feasibility or the tolerance conditions are violated.

A.5 Positive–Balance Aversion

In this section, we focus on the dynamics of the second largest tuition exchange program: The Council of Independent Colleges Tuition Exchange Program (CIC-TEP). In particular, we analyze the equilibrium of the quota revelation game in CIC-TEP. In CIC-TEP, member colleges are not suspended from the program for running a negative balance. As a consequence, all students are certified as eligible. The only requirement of the program for the member colleges is awarding at least 3 students scholarships.[7]

Since the colleges are not suspended from the program when they have a negative balance, we do not need to assume that colleges are negative–balance averse. On the other hand, holding a positive balance cannot be considered an insurance device for colleges for the continuation of their memberships. Moreover, importing more students than the number of students exported can be seen as a financial burden for colleges. Therefore, colleges may prefer not to have a positive balance. To capture the positive–balance aversion explicitly, we make the following assumption in some of our results in this section:

Positive–Balance Aversion: College c prefers all μ with $b_c^\mu = 0$ to all ν with $b_c^\nu > 0$, and otherwise ranks matchings based on its preferences over the incoming class.

[7]A college is not supposed to fulfill this requirement if the number of applicants for that college is less than 3.

Proposition 24 gives us a comparative result regarding how the balances of colleges change when they increase their import quotas.

Proposition 24 *When college c sets q_c as its import quota, suppose π is a stable matching. When it sets $\tilde{q}_c$ such that $\tilde{q}_c \geq q_c$, suppose $\tilde{\pi}$ is a stable matching. Then $b_c^{\tilde{\pi}} \in \{b_c^{\pi}, ..., b_c^{\pi} + \tilde{q}_c - q_c\}$.*

We have shown that the balances of colleges are the same in all stable matchings. In the rest of the proof we consider π and $\tilde{\pi}$ as the outcome of DA for the corresponding problems. We consider two possible cases: (1) $|\pi(c)| < q_c$ and (2) $|\pi(c)| < q_c$.

- $|\pi(c)| < q_c$ and $\tilde{q}_c = q_c + k$ for $k > 0$: Due to the non-wastefulness of π, $\pi(s) P_s c$ for all $s \in S \setminus \pi(c)$. We know that the DA mechanism is resource monotonic: when the number of seats increases then every student will be weakly better off (Kesten, 2006). That is, $\pi''(s) R_s \pi(s)$ for all $s \in S$. By combining resource monotonicity and non-wastefulness, we can say that if a student is not assigned to c in π then she will not be assigned to c in $\tilde{\pi}$. Therefore $|\tilde{\pi}(c)| < q_c$. By combining resource monotonicity and individual rationality of the DA mechanism, we can say if a student is assigned to a college in π then she will be also assigned to a college in $\tilde{\pi}$. Now we need to show that all students assigned to c in π are also assigned to c in $\tilde{\pi}$. Suppose not. Let $\pi(i) = c \neq \tilde{\pi}(i)$. Due to non-wastefulness, $\tilde{\pi}(i) P_i c$. Moreover, i should be acceptable for

$\tilde{\pi}(i)$ not to violate individual rationality of matching $\tilde{\pi}$. The reason why i is not assigned to $\tilde{\pi}(i)$ in the stable matching π is that all seats of $\tilde{\pi}(i)$ are filled by students with higher priority. Therefore, at least one of the students assigned to $\tilde{\pi}(i)$ in matching π is assigned to a better college. Denote this student by i'. Similarly, due to non-wastefulness and individual rationality, $\tilde{\pi}(i')P_{i'}c$ and $i'P_{\tilde{\pi}(i')}\emptyset$. The reason why i' is not assigned to $\tilde{\pi}(i')$ in the stable matching π is that all seats of $\tilde{\pi}(i')$ are filled by students with higher priority. Therefore, at least one of the student assigned to $\tilde{\pi}(i')$ in matching π is assigned to a better college. If we continue like this, we will have a Pareto–improving cycle without violating the priorities of students. This contradicts the fact that π is undominated stable matching. Therefore, $b_c^{\tilde{\pi}} = b_c^{\pi}$.

- $|\pi(c)| = q_c$ and $\tilde{q}_c = q_c + k$ for $k > 0$: For this case we refer to the proof of Proposition 7. In that proof we show that $b_c^{\tilde{\pi}} \in \{b_c^{\pi}, ..., b_c^{\pi} + \tilde{q}_c - q_c\}$.

Proposition 24 shows that if a college has a positive balance in a stable outcome of a problem, then increasing the import quota is never a best response for that college. Moreover, there always exists $\tilde{q}_c < q_c$ such that college c with a positive balance in a stable matching of problem $[q, e, P]$ has a non-positive balance in all stable matchings of the problem $[(\tilde{q}_c, q_{-c}), e, P]$.

In the following theorem, we show if a college has a positive balance in a stable matching when all colleges report their true import quotas then in any

Nash equilibrium of quota revelation games associated with a stable mechanism, the number of students assigned to a college is less than the number of students assigned under truth-quota reporting.

Theorem 18 *Let* $q = (q_c)_{c \in C}$ *be the vector of true import quotas and* μ *be the outcome selected by stable mechanism* ψ *in problem* $[q, e, P]$ *where* $e_c = |S_c|$ *for all* $c \in C$*. Under positive–balance aversion, if there exists a college* $c' \in C$ *such that* $b^{\mu}_{c'} > 0$*, then in any Nash equilibrium of the import quota revelation game associated with* ψ*, college* c' *reports its import quota weakly less than* $|\mu(c')| - b^{\mu}_{c'}$*.*

Since the balance of any stable matching is the same we consider ψ as the DA mechanism. Suppose the statement is not true. Let $\tilde{q} = (\tilde{q}_c)_{c \in C}$ be a Nash equilibrium profile and $\tilde{q}_{c'} > |\mu(c')| - b^{\mu}_{c'}$. First consider the case where $\tilde{q}_c = q_c$ for all $c \in C \setminus \{c'\}$. From the first part of the proof of Proposition 24, if c' sets its quota to $|\mu(c')|$, college c' will have the same balance and the assignment of all the students will not change. Then if c' keeps lowering its import quota, due to resource monotonicity, exports of college c' will never increase. But it is possible that the number of students exported by college c' may decrease as a consequence of a decrease in the import quota. Therefore, in the best case, college c' can have a non-positive (zero) balance when it sets its import quota to $|\mu(c')| - b^{\mu}_{c'}$.

Now consider the case where $\tilde{q}_c \neq q_c$ for some $c \in C \setminus \{c\}$. Due to responsive preferences, if $\tilde{q}_c > q_c$, then the number of students assigned to the college c should be less than or equal to q_c. Otherwise c can deviate and set its import quota to q_c. Therefore, the same outcome will result by setting import quota to its true value. This follows from the proof of Proposition 24. Here we will consider the under-reporting case. When some colleges under-report their import quotas due to resource monotonicity the number of students exported by college c' will weakly decrease. Moreover, due to stability, the number of students assigned to c' will not decrease. Therefore, in any stable matching of problem $[(q_{c'}, (\tilde{q})_{C\setminus\{c'\}}), e, P]$, the balance of college c' will be greater than $b^{\mu}_{c'}$. For the same reasons explained in the earlier case, c' will have a positive balance if it sets its import quota greater than $|\mu(c')| - b^{\mu}_{c'}$.

Appendix B

Chapter 2 Appendix

[Proof of Theorem 11]

We first show that the TTC mechanism satisfies all of the axioms in the theorem. Then, we show that it is the unique mechanism satisfying all of the axioms. Pareto efficiency and strategy-proofness of TTC follows from Abdulkadiroğlu and Sönmez (2003).

Mutual Best: Suppose TTC does not satisfy mutual best. Then, there exists a student school pair, (i, s), such that student i has the highest priority for school s and prefers school s to any other school and i is not assigned to s by TTC. In the first step of the TTC, s will point to i and i will point to s. They will form a cycle and i will be assigned to s. Therefore, TTC satisfies mutual best.

Resource Monotonicity for Top-Ranked Student: To show that TTC is resource monotonic for top-ranked students take a student school pair (i, s) such that $s \in t_i^{\succ}$ and $q_s > 0$. Denote the assignment of TTC in problem $[I, S, q, P, \succ]$ with μ. Now consider the problem $[I, S, (\tilde{q}_s, q_{-s}), P, \succ]$ where $\tilde{q}_s > q_s$. We consider a variant of the TTC mechanism in which only one cycle is removed in each step.[1] Fix the cycle selection rule. In particular, let $Cy(k)$ be the cycle that is selected in the k^{th} step of the variant of the TTC mechanism when we consider the problem $[I, S, q, P, \succ]$. Let s be removed in step k of TTC when we consider problem $[I, S, q, P, \succ]$. We will also select $Cy(\tilde{k})$ in step $\tilde{k} < k$ if we observe that cycle when we run the variant of TTC for the problem $[I, S, (\tilde{q}_s, q_{-s}), P, \succ]$.

School s cannot be removed before student i is assigned to a school in problem $[I, S, q, P, \succ]$. Therefore, i is assigned in step $k' \leq k$ in the problem $[I, S, q, P, \succ]$. To see this recall that in the TTC mechanism, s will point to i until i is removed. Therefore, none of the seats of s will be assigned to any student before i is removed. Also note that all the cycles selected in step $k'' < k'$ in problem $[I, S, q, P, \succ]$ will be observed in step k'' of TTC when we consider the problem $[I, S, (\tilde{q}_s, q_{-s}), P, \succ]$ because none of them includes a student pointing to s and an increase in the number of available seats in s will not affect their assignments. As a result the set of remaining schools in step k' of the TTC mechanism in both problem will be the same and we will observe the cycle including i in both problems.

Weak Consistency: We again consider the variant of the TTC that is defined above. Let J be the set of students and let $\mu(J)$ be their assignments. Due to the requirement in the definition of the weak consistency we only check the case in which each student in J has the highest priority for one of the

[1]TTC is independent of the order in which cycles are selected.

schools in $\mu(J)$. Suppose none of the students in J belongs to a $Cy(k)$ where $k < \tilde{k}$. Then, it is clear that the assignment of students in $Cy(k)$ where $k < \tilde{k}$ will not be affected by the removal of students in J with their assignments. Suppose $i \in Cy(\tilde{k})$. Let $\mu(i)$ be his assignment. Therefore, i_1 who is the top-ranked student in the priority order of $\mu(i)$ should be in J. This is also true for the top-ranked student of the school that i_1 is assigned. Due to the finiteness we should have a cycle. That is, $Cy(\tilde{k}) \subseteq J$ and $\mu(Cy(\tilde{k})) \subseteq \mu(J)$. Therefore, removing these students before running the TTC mechanism or removing them within the mechanism will not affect the assignments of the remaining students.

Uniqueness: Suppose there exists another mechanism ϕ satisfying all these 5 properties and there exists a problem $[I, S, q, P, \succ]$ in which ϕ and TTC select different matchings. Let $TTC\,[I, S, q, P, \succ] = \mu$ and $\phi\,[I, S, q, P, \succ]$ be the outcome of TTC and ϕ in problem $[I, S, q, P, \succ]$, respectively. We will consider the version of TTC mechanism in which only one cycle is removed in a step and if there are more than 1 cycle the one which will be removed is selected based on some exogenous rule. Then suppose that each student removed before step $k \geq 1$ of the TTC mechanism is assigned to the same school under ϕ and TTC. Denote these students with set J. That is, $TTC\,[I, S, q, P, \succ]\,(j) = \phi\,[I, S, q, P, \succ]\,(j)$ for all $j \in J$. Let i be the student who is removed in the step k of TTC and assigned to a different school by ϕ. If we remove students assigned in the first step of TTC with their assignments then we get the reduced problem $[I^1, S^1, q^1, P^1, \succ^1]$ where $I^1 = I \backslash Cy(1)$,

$S^1 = S$, $q_s^1 = q_s - \sum\limits_{j \in Cy(1)} 1(\phi[I,S,q,P,\succ](j) = s)$, $P^1 = P_{I^1}$ and $\succ^1 = \succ^{I^1}$. Note that each student $h \in Cy(1)$ $|t_h^{\succ} \cap TTC[I,S,q,P,\succ]| = |t_h^{\succ} \cap \phi[I,S,q,P,\succ]| = 1$. Due to weak consistency assignments of the students in I^1 in the outcome of both mechanisms will not change. That is, $\phi[I^1,S^1,q^1,P^1,\succ^1](i') = \phi[I,S,q,P,\succ](i')$ and $TTC[I^1,S^1,q^1,P^1,\succ^1](i') = TTC[I,S,q,P,\succ](i')$ for all $i' \in I^1$. Moreover, $\phi[I^1,S^1,q^1,P^1,\succ^1](j) = TTC[I^1,S^1,q^1,P^1,\succ^1](j)$ for all $j \in J \cap I^1$. If $k \neq 2$, then we remove students in $Cy(2)$ with their assignments in $TTC[I^1,S^1,q^1,P^1,\succ^1]$. We get the reduced problem $[I^2,S^2,q^2,P^2,\succ^2]$ where $I^2 = I^1 \setminus Cy(2)$, $S^2 = S$, $q_s^2 = q_s^1 - \sum\limits_{j \in Cy(2)} 1(TTC[I^1,S^1,q^1,P^1,\succ^1](j) = s)$, $P^2 = P_{I^2}$ and $\succ^2 = \succ^{I^2}$. Note that for each student $h \in Cy(1)$ $|t_h^{\succ^1} \cap TTC[I^1,S^1,q^1,P^1,\succ^1]| = |t_h^{\succ^1} \cap \phi[I^1,S^1,q^1,P^1,\succ^1]| = 1$. Note that each student $h \in Cy(2)$ $|t_h^{\succ^1} \cap TTC[I^1,S^1,q^1,P^1,\succ^1]| = |t_h^{\succ^1} \cap \phi[I^1,S^1,q^1,P^1,\succ^1]| = 1$. Due to weak consistency assignments of the students in I^2 in the outcome of both mechanisms will be the same in problem $[I^1,S^1,q^1,P^1,\succ^1]$ and $[I^2,S^2,q^2,P^2,\succ^2]$. That is, $\phi[I^2,S^2,q^2,P^2,\succ^2](i') = \phi[I^1,S^1,q^1,P^1,\succ^1](i')$ and $TTC[I^2,S^2,q^2,P^2,\succ^2](i') = TTC[I^1, S^1, q^1, P^1, \succ^1](i')$ for all $i' \in I^2$. Moreover, $\phi[I^2,S^2,q^2,P^2,\succ^2](j) = TTC[I^2,S^2,q^2,P^2,\succ^2](j)$ for all $j \in J \cap I^2$. Similarly, we can continue removing students with their assignments in the following order: $Cy(3) - ... - Cy(k-2) - Cy(k-1)$. Let $I^{k'} = I^{k'-1} \setminus Cy(k')$, $S^{k'} = S$, $q_s^{k'} = q_s^{k'-1} - \sum\limits_{j \in Cy(k')} 1(TTC[I^{k'-1},S^{k'-1},q^{k'-1},P^{k'-1},\succ^{k'-1}](j) = s)$, $P^{k'} = P_{I^{k'}}$ and $\succ^{k'} = \succ^{I^{k'}}$ where $k' < k$. Note that

$$q_s^{k'} = q_s^{k'-1} - \sum_{j \in Cy(k')} 1(\phi[I^{k'-1}, S^{k'-1}, q^{k'-1}, P^{k'-1}, \succ^{k'-1}](j) = s)$$

and for each student $h \in Cy(k')$

$$|t_h^{\succ^{k'-1}} \cap TTC[I^{k'-1}, S^{k'-1}, q^{k'-1}, P^{k'-1}, \succ^{k'-1}]| = 1.$$

$$|t_h^{\succ^{k'-1}} \cap \phi[I^{k'-1}, S^{k'-1}, q^{k'-1}, P^{k'-1}, \succ^{k'-1}]| = 1.$$

Due to weak consistency

$$\phi[I^{k'}, S^{k'}, q^{k'}, P^{k'}, \succ^{k'}](i') = \phi[I^{k'-1}, S^{k'-1}, q^{k'-1}, P^{k'-1}, \succ^{k'-1}](i')$$

and

$$TTC[I^{k'}, S^{k'}, q^{k'}, P^{k'}, \succ^{k'}](i') = TTC[I^{k'-1}, S^{k'-1}, q^{k'-1}, P^{k'-1}, \succ^{k'-1}](i')$$

for all $i' \in I^{k'}$ where $k' < k$. Denote reduced problem $[I^{k-1}, S^{k-1}, q^{k-1}, P^{k-1}, \succ^{k-1}$ $]$ with $\left[\tilde{I}, \tilde{S}, \tilde{q}, \tilde{P}, \tilde{\succ}\right]$ where $\tilde{I} = I \setminus J$, $\tilde{S} = S$, $\tilde{q}_s = q_s - \sum_{i \in J} 1(TTC[I, S, q, P, \succ$ $](i) = s) = q_s - \sum_{i \in J} 1(\phi[I, S, q, P, \succ](i) = s)$, $\tilde{P} = P_{\tilde{I}}$ and $\tilde{\succ} = \succ^{\tilde{I}}$. We apply TTC and ϕ to the problem $\left[\tilde{I}, \tilde{S}, \tilde{q}, \tilde{P}, \tilde{\succ}\right]$. Due to weak consistency $TTC[\tilde{I}, \tilde{S}, \tilde{q}, \tilde{P}, \tilde{\succ}](i) = TTC[I, S, q, P, \succ](i)$, $\phi[I, S, q, P, \succ](i) = \phi[\tilde{I}, \tilde{S}, \tilde{q}, \tilde{P}, \tilde{\succ}](i)$ and $TTC[\tilde{I}, \tilde{S}, \tilde{q}, \tilde{P}, \tilde{\succ}](i) \neq \phi[\tilde{I}, \tilde{S}, \tilde{q}, \tilde{P}, \tilde{\succ}](i)$. When we apply TTC mechanism to the problem $\left[\tilde{I}, \tilde{S}, \tilde{q}, \tilde{P}, \tilde{\succ}\right]$ we remove the cycles one by one by using the same cycle selection rule used in problem $[I, S, q, P, \succ]$. In the reduced problem student i will be removed in the first step of the TTC mechanism. Let s be the school pointing to i in the first step of TTC mechanism in the reduced problem $\left[\tilde{I}, \tilde{S}, \tilde{q}, \tilde{P}, \tilde{\succ}\right]$. Note that there can be more than one school pointing to i in the first step of TTC mechanism. Here school s is the one which is in the cycle together with i. By the definition of the TTC mechanism student i should be the top-ranked student in $\tilde{\succ}_s$. By the definition of TTC, TTC assigns i to his favorite school[2] among the ones with available seats in problem $[\tilde{I}, \tilde{S}, \tilde{q}, \tilde{P}, \tilde{\succ}]$. Therefore, $TTC[\tilde{I}, \tilde{S}, \tilde{q}, \tilde{P}, \tilde{\succ}](i)\tilde{P}_i\phi[\tilde{I}, \tilde{S}, \tilde{q}, \tilde{P}, \tilde{\succ}](i)$. We consider two cases. In the first case TTC assigns student i to s and in the second case TTC assigns i to another school.

Case 1: Student i points to the school s in the first step of TTC in the reduced problem $[\tilde{I}, \tilde{S}, \tilde{q}, \tilde{P}, \tilde{\succ}]$. School s should be the most preferred school in $\tilde{P}_i$ among the ones having available seats. Suppose i reports $P_i' : sP_i's_\emptyset$. Due to the strategy-proofness TTC will assign i to s and ϕ will assign i to $s_\emptyset$. Any mutually best mechanism should assign i to s in the reduced problem. Therefore, ϕ fails to satisfy mutual best.

Case 2: In this case i is assigned to $s' \neq s$. Since s belongs to the cycle, there is another student $i_1 \in Cy(k)$ assigned to school s. Now suppose student i reports $s'P_i'sP_i's_\emptyset$. TTC will select the same matching in problems $[\tilde{I}, \tilde{S}, \tilde{q}, \tilde{P}, \tilde{\succ}]$ and $[\tilde{I}, \tilde{S}, \tilde{q}, (P_i', \tilde{P}_{-i}), \tilde{\succ}]$. Due to the strategy-proofness ϕ will as-

[2] Here we consider preference profile $\tilde{P}_i$. This statement is also true for P_i since $\tilde{P}_i = P_i$.

sign i to either s where he is top-ranked or $s_\emptyset$ in problem $[\tilde{I}, \tilde{S}, \tilde{q}, (P'_i, \tilde{P}_{-i}), \tilde{\succ}]$.[3] First consider the latter case where $\phi[\tilde{I}, \tilde{S}, \tilde{q}, (P'_i, \tilde{P}_{-i}), \tilde{\succ}](i) = s_\emptyset$. Now consider the problem $[\tilde{I}, \tilde{S}, \tilde{q}, (P''_i, \tilde{P}_{-i}), \tilde{\succ}]$ in which i submits $sP''_i s_\emptyset$. Due to strategy-proofness $\phi[\tilde{I}, \tilde{S}, \tilde{q}, (P''_i, \tilde{P}_{-i}), \tilde{\succ}](i) = s_\emptyset$. However this violates mutual best. Therefore the latter case is not possible. Therefore, when i submits P'_i he will be assigned to s by ϕ in problem $[\tilde{I}, \tilde{S}, \tilde{q}, (P'_i, \tilde{P}_{-i}), \tilde{\succ}]$.

Now consider the case where i submits P'_i and school s have only one available seat. That is, we are considering problem $[\tilde{I}, \tilde{S}, \bar{q}^0, \bar{P}^0, \tilde{\succ}]$ where $\bar{q}^0_{s''\neq s} = \tilde{q}_{s''\neq s}$, $\bar{q}^0_s = 1$, $\bar{P}^0_{i'\neq i} = \tilde{P}_{i'\neq i}$ and $\bar{P}^0_i = P'_i$. Then, $TTC[\tilde{I}, \tilde{S}, \bar{q}^0, \bar{P}^0, \tilde{\succ}] = s'$. Suppose $\phi[\tilde{I}, \tilde{S}, \bar{q}^0, \bar{P}^0, \tilde{\succ}](i) = s$'. We showed that $\phi[\tilde{I}, \tilde{S}, \tilde{q}, \bar{P}, \tilde{\succ}](i) = s$. If $\tilde{q}_s = \bar{q}_s$, then mechanism ϕ selects two different outcome for the same problem. If $\tilde{q}_s > \bar{q}_s$, then resource monotonicity for top-ranked students is violated since i becomes worse off when the number of available seats in school s increases. Then he will be assigned another school or $s_\emptyset$. Due to the aforementioned reasons he will be assigned to s. Therefore, student i_1 who is assigned to s in $TTC[\tilde{I}, \tilde{S}, \bar{q}^0, \bar{P}^0, \tilde{\succ}]$ will be assigned to another school by ϕ. Given s is the top choice of i_1 among the schools with available seats i_1 prefers his assignment in $TTC[\tilde{I}, \tilde{S}, \bar{q}^0, \bar{P}^0, \tilde{\succ}]$ to $\phi[\tilde{I}, \tilde{S}, \bar{q}^0, \bar{P}^0, \tilde{\succ}]$.

Since student i_1 is assigned to a school by TTC in the first step of problem $[\tilde{I}, \tilde{S}, \bar{q}^0, \bar{P}^0, \tilde{\succ}]$ there should be another school s_1 where i_1 is the top-ranked

[3]Here, it is possible that i can be also assigned to another school that he doesn't include to his preference list. However, we can prove that this will violate either strategy-proofness or mutual best as a similar way that we follow for showing that i cannot be assigned to $s_\emptyset$.

student according to $\tilde{\succ}$. Then consider the following problem $[\tilde{I}, \tilde{S}, \bar{q}^1, \bar{P}^1, \tilde{\succ}]$ where $\bar{q}^1_{s'' \neq s_1} = \bar{q}_{s'' \neq s_1}$, $\bar{q}^1_{s_1} = 1$, $\bar{P}^1_{i' \neq i_1} = \bar{P}_{i' \neq i_1}$ and $\bar{P}^1_{i_1} : s\bar{P}^1_{i_1} s_1 \bar{P}^1_{i_1} s_\emptyset$. Similarly we can show that $TTC[\tilde{I}, \tilde{S}, \bar{q}^1, \bar{P}^1, \tilde{\succ}](i_1) = s$ and $\phi[\tilde{I}, \tilde{S}, \bar{q}^1, \bar{P}^1, \tilde{\succ}](i_1) = s_1$. By following similar steps we get problem $[\tilde{I}, \tilde{S}, \bar{q}, \bar{P}, \tilde{\succ}]$ where $\bar{q}_{s'' \notin \mu(Cy(k))} = q_{s'' \notin \mu(Cy(k))}$, $\bar{q}_{s'' \in \mu(Cy(k))} = 1$, $\bar{P}_{i' \notin Cy(k)} = P_{i' \notin Cy(k)}$, $\bar{P}_{i_x \in Cy(k)} : \mu(i_x)\bar{P}_{i_x} s_x \bar{P}_{i_x} s_\emptyset$, $s_x \in \mu(Cy(k))$ and student i_x is the top ranked student in $\tilde{\succ}_{s_x}$. In this problem, $TTC[\tilde{I}, \tilde{S}, \bar{q}, \bar{P}, \tilde{\succ}](i_x) = \mu(i_x)$ and $\phi[\tilde{I}, \tilde{S}, \bar{q}, \bar{P}, \tilde{\succ}](i_x) = s_x$. Therefore they will be assigned to strictly worse school by ϕ and no other student will be assigned to those schools since all schools quota will be equalized to 1. Therefore a trade between these students will increase the welfare without worsening any other student and ϕ fails to be Pareto-efficient.

Appendix C

Appendix of Chapter 3

The two mechanisms do not always select different matchings for a given house allocation problem. In the following proposition we state the conditions that guarantee the equivalence of the outcomes of the DA and TTC mechanisms.

Proposition 25 *If* $0 < |I_E| \leq 2$ *and each existing agent has either the top rank or the second rank in* f*, then TTC and DA will select the same matching for all problems.*

Let $|I_E| = 2$, $f(1) = i$ and $f(2) = j$ are both in I_E. We use the same variants of TTC and DA explained in the proof of Proposition 1. We look at the following cases.

Case 1: If i prefers a new house or h_i most, then he will get that house in both π and μ. Then j will have top priority in all remaining houses and gets his most preferred house among the remaining ones. Then the outcome for the remaining new students will be exactly the same in $\mu$and π.

Case 2: If i prefers h_j most, i will get h_j in both π and μ as long as j does not prefer h_j most. In this case everything mentioned in case 1 will hold.

Otherwise, j will be assigned to h_j and i will get his second choice in both π and μ.

Let $|I_E| = 1$. Let existing agent i have the highest priority in f. He will have the highest priority for any house he prefers most and will get that house in both π and μ. Then the outcome for the remaining new students will be exactly the same in μ and π. Let $i = f(2)$. In this case $f(1) \in I_N$ will get his top choice unless his top choice is not h_i and i does not prefer h_i most. This corresponds to case 2 described above.

The conditions mentioned in Proposition 25 are the necessary conditions for the equivalence of DA and TTC in the house allocation problem. We will illustrate this in the following examples. In the first example we show that if $|I_E| > 2$, then there exists a problem in which DA and TTC select different outcomes.

Example 12 *There are 3 houses $H = \{h_{i_1}, h_{i_2}, h_{i_3}\}$ and 3 students $I = \{i_1, i_2, i_3\}$. Here i_1, i_2 and i_3 are existing students currently occupying h_{i_1}, h_{i_3} and h_{i_3}, respectively. Let $f(1) = i_1$, $f(2) = i_2$ and $f(3) = i_3$. The preference profile is given by: $h_{i_3} P_{i_1} h_{i_1} P_{i_1} h_{i_2}$, $h_{i_1} P_{i_2} h_{i_2} P_{i_2} h_{i_3}$ and $h_{i_1} P_{i_3} h_{i_3} P_{i_3} h_{i_2}$.*

In this problem TTC will select the following matching: $\pi(i_1) = h_{i_3}$, $\pi(i_2) = h_{i_2}$ and $\pi(i_3) = h_{i_1}$. On the other hand DA will select the following matching: $\mu(i_1) = h_{i_1}$, $\mu(i_2) = h_{i_2}$ and $\mu(i_3) = h_{i_3}$.

In the following example we show that when $0 < |I_E| \leq 2$, if each existing agent does not have either the top rank or the second rank in f, then

TTC and DA will select different matchings.

Example 13 *In this example* $|I_E| = 1$*, but it is easy to construct an example where* $|I_E| = 2$*. For instance, we can consider the same example where* i_1 *is the existing tenant of* h_1*.*

There are 3 houses $H = \{h_1, h_2, h_{i_3}\}$ *and 3 students* $I = \{i_1, i_2, i_3\}$*. Here* i_3 *is an existing student currently occupying* h_{i_3}*. Let* $f(1) = i_1$*,* $f(2) = i_2$ *and* $f(3) = i_3$*. The preference profile is given by:* $h_{i_3} P_{i_1} h_1 P_{i_1} h_2$*,* $h_1 P_{i_2} h_2 P_{i_2} h_{i_3}$ *and* $h_1 P_{i_3} h_{i_3} P_{i_3} h_2$*.*

In this problem TTC selects the following matching: $\pi(i_1) = h_{i_3}$*,* $\pi(i_2) = h_2$ *and* $\pi(i_3) = h_1$*. On the other hand DA selects the following matching:* $\mu(i_1) = h_1$*,* $\mu(i_2) = h_2$ *and* $\mu(i_3) = h_{i_3}$*.*

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